Quiz 4

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Quiz 4 – problem1

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$

$$if\ a > 0$$

$$F_Y(y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = \frac{dF_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$if\ a < 0$$

$$F_Y(y) = P\left(X \ge \frac{y-b}{a}\right) = 1 - F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow f_Y(y) = -\frac{dF_X\left(\frac{y-b}{a}\right)}{d\left(\frac{y-b}{a}\right)} \cdot \frac{d\left(\frac{y-b}{a}\right)}{dy} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad \therefore f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

Quiz 4 – *problem2*

- Let X be a random variable such that
  $M_X(s) = ae^{2s} + be^{4s} + ce^{8s}$, $E[X] = 4$ and $var(X) = 6$
  Find a , b , c , and the PMF of X .

$$\Rightarrow \because E[X] = 2a + 4b + 8c = 4$$

$$var(X) = 4a + 16b + 64c - (E[X])^2 = 6$$

$$\therefore \begin{cases} a + b + c = 1 \\ 2a + 4b + 8c = 4 \\ 4a + 16b + 64c - 16 = 6 \end{cases}$$

$$a = \frac{1}{2} \ , \ b = \frac{1}{4} \ , \ c = \frac{2}{8} = \frac{1}{4} \ , \ p_X(x) = \begin{cases} \frac{1}{2}, if \ x = 2 \\ \frac{1}{4}, if \ x = 4 \\ \frac{1}{4}, if \ x = 8 \end{cases}$$

Quiz 4 – problem3-(i)

- (i) $f_X(x) = \lambda e^{-\lambda x}\ , x \geq 0$

$$M_X(s) = \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(s-\lambda)x} dx$$

$$= \lambda \left. \frac{e^{(s-\lambda)x}}{(s-\lambda)} \right|_0^\infty \quad , if\ s - \lambda < 0$$

$$= \frac{\lambda}{\lambda - s}$$

Quiz 4 – *problem3-(ii)&(iii)*

- (ii)

$$if\ Y = aX + b$$

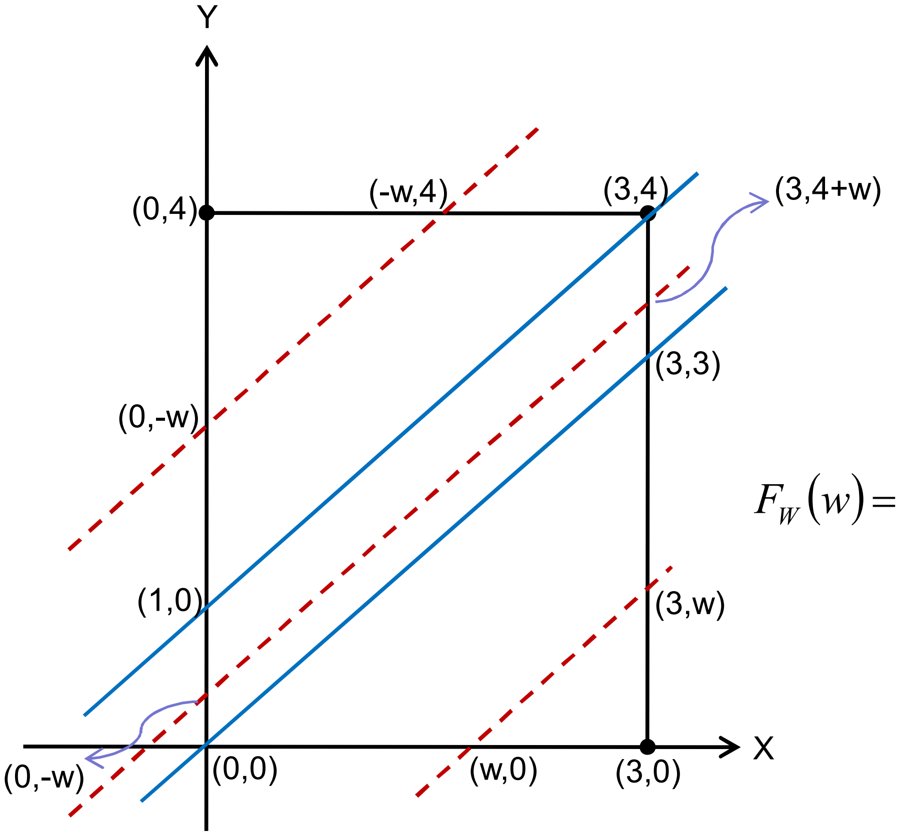
$$M_Y(s) = E\left[e^{s(aX+b)}\right] = e^{sb}E\left[e^{saX}\right] = e^{sb}M_X(sa)$$

$$\therefore M_Y(s) = e^{3s}M_X(2s) = e^{3s}\frac{\lambda}{\lambda - 2s}$$

- (iii)

$$M_W(s) = M_X(2s)M_Z(s) = \frac{\lambda}{\lambda - 2s}\frac{\eta}{\eta - s}$$

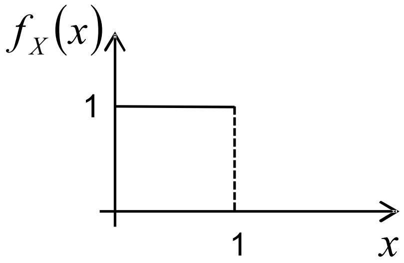
Quiz 4 – problem4-(i)

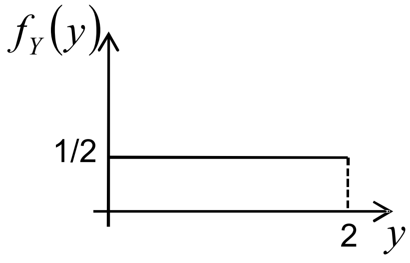


$$F_W(w)=\begin{cases}\frac{1}{12}\left[\frac{1}{2}(4+w)^2\right]=\frac{(4+w)^2}{24} & ,if\ -4\le w\le -1\\ \frac{1}{12}\left\{\frac{[3+(1+w)]+(1+w)}{2}\cdot 3\right\}=\frac{5+2w}{8} & ,if\ -1\le w\le 0\\ \frac{1}{12}\left[12-\frac{1}{2}(3-w)^2\right]=1-\frac{(3-w)^2}{24} & ,if\ 0\le w\le 3\end{cases}$$

$$\therefore f_W(w)=\begin{cases}\frac{1}{3}+\frac{w}{12} & ,if\ -4\le w\le -1\\ \frac{1}{4} & ,if\ -1\le w\le 0\\ \frac{1}{4}-\frac{w}{12} & ,if\ 0\le w\le 3\end{cases}$$

Quiz 4 – problem4-(ii)

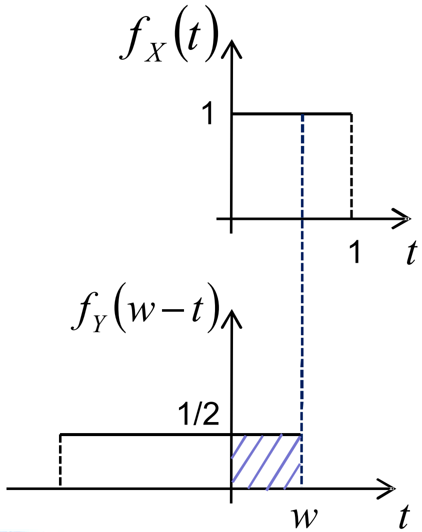




$$f_W(w) = \int_0^3 f_X(t) f_Y(w-t)\,dt$$

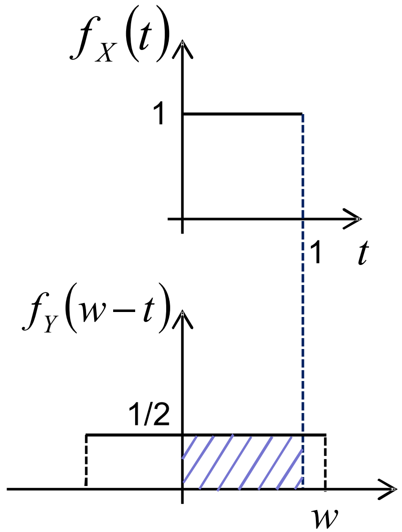
$$\text{if } 0 \le w \le 1$$

$$f_W(w) = \frac{w}{2}$$



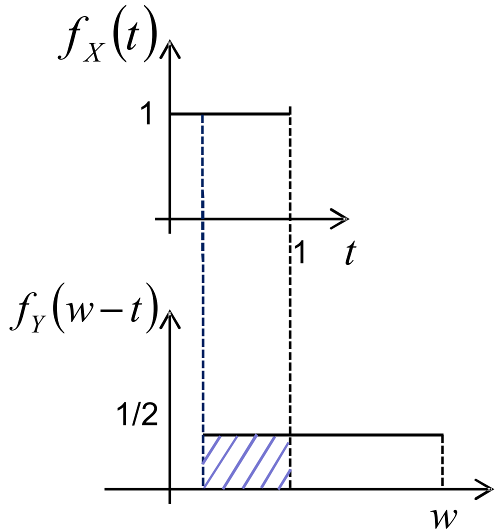
$$\text{if } 1 \le w \le 2$$

$$f_W(w) = \frac{1}{2}$$



$$\text{if } 2 \le w \le 3$$

$$f_W(w) = \frac{1}{2}\left(1 - (w-2)\right) = \frac{3-w}{2}$$



Quiz 4 – problem5

$$\because E[X \mid Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\,dx = \int_{0}^{\frac{2-y}{2}} x \frac{2}{2-y}\,dx = \frac{2-y}{4}$$

$$\therefore E[X] = \frac{2}{4} - \int_{0}^{2} \frac{y}{4}\frac{2-y}{2}\,dy = \frac{1}{2} - \left[\frac{4}{8} - \frac{8}{24}\right] = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$