

Eigenvalues

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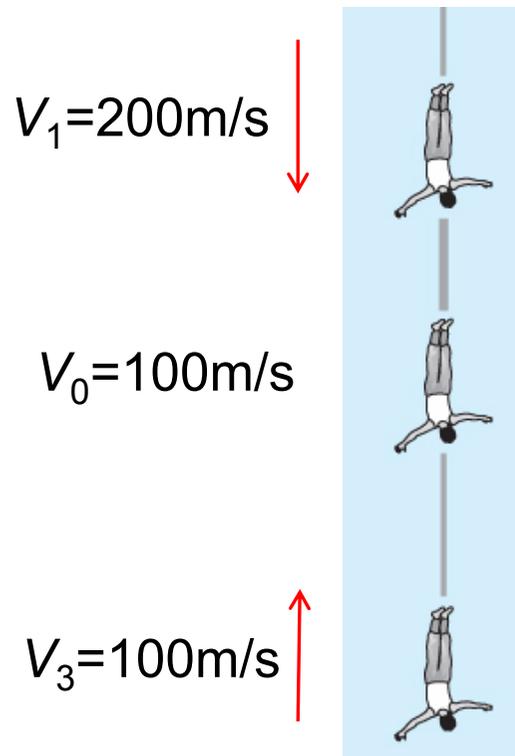
Reference:

1. *Applied Numerical Methods with MATLAB for Engineers*, Chapter 13 & Teaching material

Chapter Objectives

- Understanding the mathematical definition of eigenvalues and eigenvectors
- Understanding the physical interpretation of eigenvalues and eigenvectors within the context of engineering systems that vibrate or oscillate
- Knowing how to implement the polynomial method
- Knowing how to implement the power method to evaluate the largest and smallest eigenvalues and their respective eigenvectors
- Knowing how to use and interpret MATLAB's eig function

Dynamics of Three Coupled Bungee Jumpers in Time



Initial Conditions

(set the jumpers' initial positions to the equilibrium values)

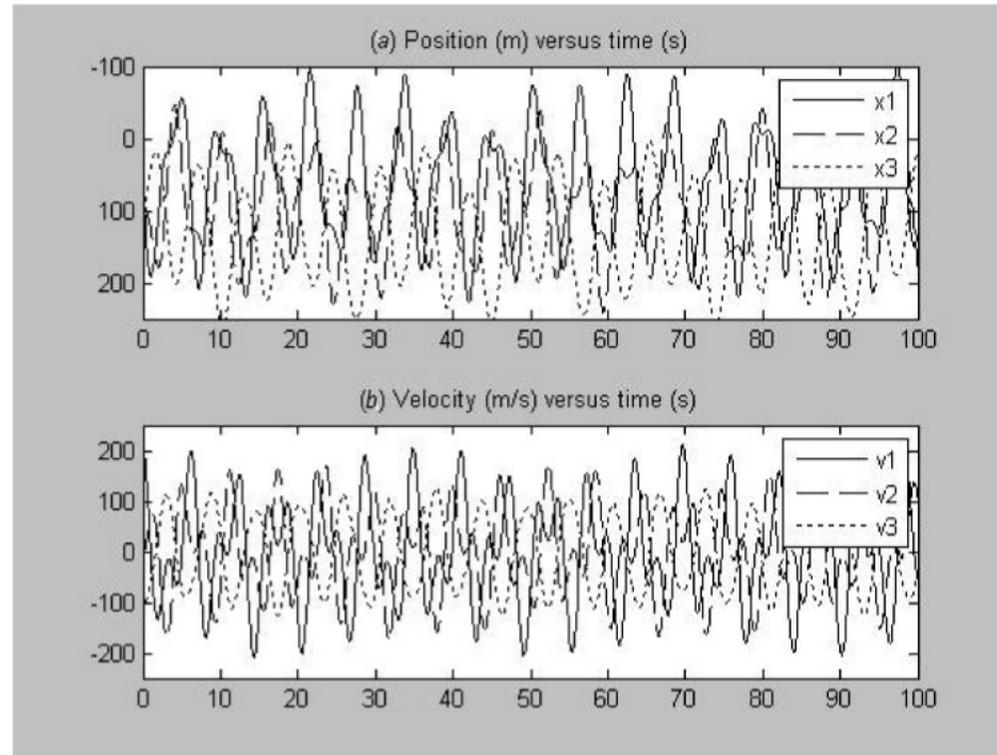
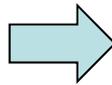


FIGURE 13.1

The (a) positions and (b) velocities versus time for the system of three interconnected bungee jumpers from Example 8.2.

- Is there an underlying (latent) pattern???

Mathematics (1/2)

- Up until now, **heterogeneous** systems:

$$[A] \{x\} = \{b\}$$

Have a unique solution when equations are linearly independent (i.e., A has a nonzero determinant)

- What about **homogeneous** systems?

$$[A] \{x\} = 0$$

- At face value, it has the trivial solution:

$$\{x\} = 0$$

- Is there another way of formulating the system so that the solution would be meaningful (nontrivial) ???

Mathematics (2/2)

- What about a homogeneous system like:

$$(a_{11} - \lambda) x_1 + a_{12} x_2 + a_{13} x_3 = 0$$

$$a_{21} x_1 + (a_{22} - \lambda) x_2 + a_{23} x_3 = 0$$

$$a_{31} x_1 + a_{32} x_2 + (a_{33} - \lambda) x_3 = 0$$

- Or, in matrix form

$$[[A] - \lambda [I]] \{x\} = 0$$

- For this case, there could be a value of λ that makes the equations equal zero. This is called an **eigenvalue**
 - For non-trivial solutions to be possible

$$|[A] - \lambda [I]| = 0$$

- Expand the determinant yields a polynomial in λ , called the **characteristic polynomial**
- The roots of the polynomial are **eigenvalues** of A

A Two-Equation Case

In order to better understand these concepts, it is useful to examine the two-equation case,

$$\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 &= 0\end{aligned}\tag{13.5}$$

Expanding the determinant of the coefficient matrix gives

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda - a_{12}a_{21}\tag{13.6}$$

which is the *characteristic polynomial*. The quadratic formula can then be used to solve for the two eigenvalues:

$$\begin{aligned}\lambda_1 &= \frac{(a_{11} - a_{22})^2 \pm \sqrt{(a_{11} - a_{22})^2 - 4a_{12}a_{21}}}{2} \\ \lambda_2 &= \frac{(a_{11} - a_{22})^2 \pm \sqrt{(a_{11} - a_{22})^2 - 4a_{12}a_{21}}}{2}\end{aligned}\tag{13.7}$$

These are the values that solve Eq. (13.5). Before proceeding, let's convince ourselves that this approach (which, by the way, is called the *polynomial method*) is correct.

The Polynomial Method (1/3)

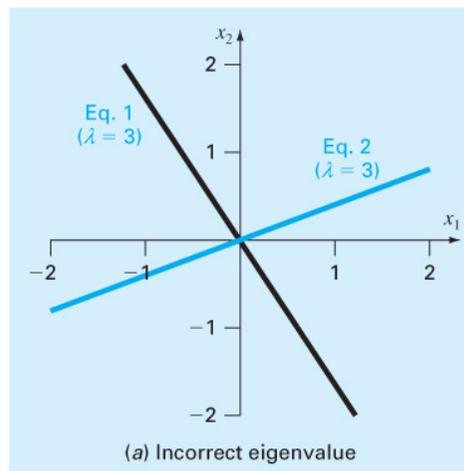
Problem Statement. Use the polynomial method to solve for the eigenvalues of the following homogeneous system:

$$\begin{aligned}(10 - \lambda)x_1 - 5x_2 &= 0 \\ -5x_1 + (10 - \lambda)x_2 &= 0\end{aligned}$$

Solution. Before determining the correct solution, let's first investigate the case where we have an incorrect eigenvalue. For example, if $\lambda = 3$, the equations become

$$\begin{aligned}7x_1 - 5x_2 &= 0 \\ -5x_1 + 7x_2 &= 0\end{aligned}$$

Plotting these equations yields two straight lines that intersect at the origin (Fig. 13.2a). Thus, the only solution is the trivial case where $x_1 = x_2 = 0$.



Example 13.1

The Polynomial Method (2/3)

To determine the correct eigenvalues, we can expand the determinant to give the characteristic polynomial:

$$\begin{vmatrix} 10 - \lambda & -5 \\ -5 & 10 - \lambda \end{vmatrix} = \lambda^2 - 20\lambda + 75$$

which can be solved for

$$\lambda_1 = \frac{20 + \sqrt{20^2 - 4(1)75}}{2} = 15, 5$$
$$\lambda_2 = \frac{20 - \sqrt{20^2 - 4(1)75}}{2} = 15, 5$$

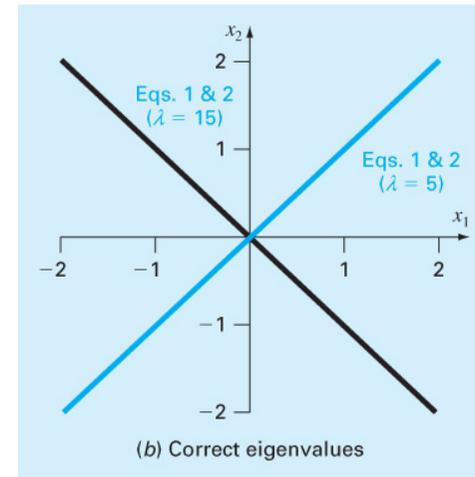
Therefore, the eigenvalues for this system are 15 and 5.

We can now substitute either of these values back into the system and examine the result. For $\lambda_1 = 15$, we obtain

$$-5x_1 - 5x_2 = 0$$

$$-5x_1 - 5x_2 = 0$$

Thus, a correct eigenvalue makes the two equations identical (Fig. 13.2b). In essence as we move towards a correct eigenvalue the two lines rotate until they lie on top of each other.



Example 13.1

The Polynomial Method (3/3)

Mathematically, this means that there are an infinite number of solutions. But solving either of the equations yields the interesting result that all the solutions have the property that $x_1 = -x_2$. Although at first glance this might appear trivial, it's actually quite interesting as it tells us that the ratio of the unknowns is a constant. This result can be expressed in vector form as

$$\{x\} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

which is referred to as the eigenvector corresponding to the eigenvalue $\lambda = 15$.

In a similar fashion, substituting the second eigenvalue, $\lambda_2 = 5$, gives

$$5x_1 - 5x_2 = 0$$

$$-5x_1 + 5x_2 = 0$$

Again, the eigenvalue makes the two equations identical (Fig. 13.2b) and we can see that the solution for this case corresponds to $x_1 = x_2$, and the eigenvector is

$$\{x\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

The eigenvectors provide the ratios of the unknowns representing the solution.

Example 13.1

MATLAB Built-in Functions

We should recognize that MATLAB has built-in functions to facilitate the polynomial method. For Example 13.1, the `poly` function can be used to generate the characteristic polynomial as in

```
>> A = [10 -5; -5 10];  
>> p = poly(A)  
  
p =  
    1   -20    75
```

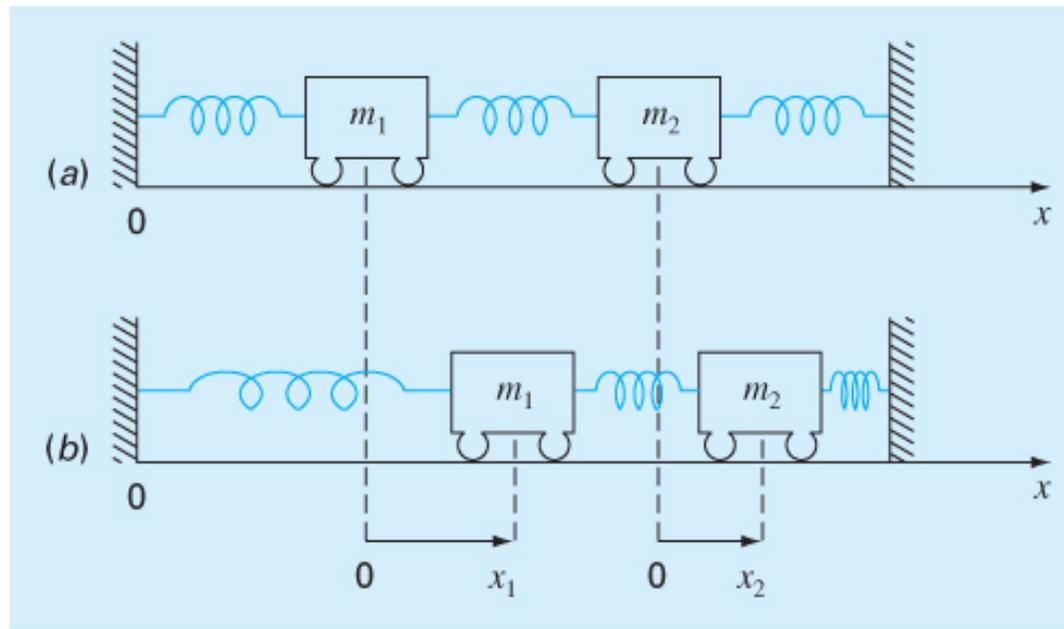
Then, the `roots` function can be employed to compute the eigenvalues:

```
>> d = roots(p)  
  
d =  
    15  
     5
```

Physical Background: Oscillations or *Vibrations* of Mass-Spring Systems

FIGURE 13.3

A two mass–three spring system with frictionless rollers vibrating between two fixed walls. The position of the masses can be referenced to local coordinates with origins at their respective equilibrium positions (a). As in (b), positioning the masses away from equilibrium creates forces in the springs that on release lead to oscillations of the masses.



Model with Force Balances

(AKA: $F = ma$)

$$m_1 \frac{d^2x_1}{dt^2} = -kx_1 + k(x_2 - x_1)$$

$$m_2 \frac{d^2x_2}{dt^2} = -k(x_2 - x_1) - kx_2$$

- Collect terms:

$$m_1 \frac{d^2x_1}{dt^2} - k(-2x_1 + x_2) = 0$$

$$m_2 \frac{d^2x_2}{dt^2} - k(x_1 - 2x_2) = 0$$

Assume a Sinusoidal Solution (1/2)

- Based on vibration theory

$$x_i = X_i \sin(\omega t)$$

amplitude

where $\omega = \frac{2\pi}{T_p}$

angular frequency

- Differentiate twice:

$$x_i'' = -X_i \omega^2 \sin(\omega t)$$

- Substitute back into system and collect terms

Assume a Sinusoidal Solution (2/2)

$$\begin{aligned} \left[\frac{2k}{m_1} - \omega^2 \right] X_1 - \frac{k}{m_1} X_2 &= 0 \\ \frac{k}{m_2} X_1 + \left[\frac{2k}{m_2} - \omega^2 \right] X_2 &= 0 \end{aligned}$$

Given: $m_1 = m_2 = 40$ kg; $k = 200$ N/m

$$\begin{aligned} (10 - \omega^2) X_1 - 5 X_2 &= 0 \\ -5 X_1 + (10 - \omega^2) X_2 &= 0 \end{aligned}$$

- This is now a homogeneous system where the eigenvalue represents the square of the angular frequency ($\lambda = \omega^2$)

Solution: The Polynomial Method

$$\begin{bmatrix} 10 - \omega^2 & -5 \\ -5 & 10 - \omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- Evaluate the determinant to yield a polynomial

$$\begin{vmatrix} 10 - \omega^2 & -5 \\ -5 & 10 - \omega^2 \end{vmatrix} = (\omega^2)^2 - 20\omega^2 + 75 = 0$$

- The two roots of this "characteristic polynomial" are the system's eigenvalues:

$$\omega^2 = \frac{15}{5} \quad \text{or} \quad \omega = \begin{matrix} 3.873 \text{ Hz} \\ 2.36 \text{ Hz} \end{matrix}$$

Interpretation

$$\omega^2 = 5 \text{ /s}^2$$

$$\omega = 2.236 \text{ /s}$$

$$T_p = 2\pi/2.236 = 2.81 \text{ s}$$

$$(10 - \omega^2) X_1 -$$

$$- 5 X_1 + (10 - \omega^2) X_2 = 0$$

$$(10 - 5) X_1 - 5 X_2 = 0$$

$$- 5 X_1 + (10 - 5) X_2 = 0$$

$$5 X_1 - 5 X_2 = 0$$

$$- 5 X_1 + 5 X_2 = 0$$

$$X_1 = X_2$$

$$V = \begin{bmatrix} -0.7071 \\ -0.7071 \end{bmatrix} \text{ eigenvector}$$

$$\omega^2 = 15 \text{ /s}^2$$

$$\omega = 3.873 \text{ /s}$$

$$T_p = 2\pi/3.373 = 1.62 \text{ s}$$

$$5 X_2 = 0$$

$$- 5 X_1 + (10 - \omega^2) X_2 = 0$$

$$(10 - 15) X_1 - 5 X_2 = 0$$

$$- 5 X_1 + (10 - 15) X_2 = 0$$

$$- 5 X_1 - 5 X_2 = 0$$

$$- 5 X_1 - 5 X_2 = 0$$

$$X_1 = -X_2$$

$$V = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix} \text{ eigenvector}$$

Principle Modes of Vibration

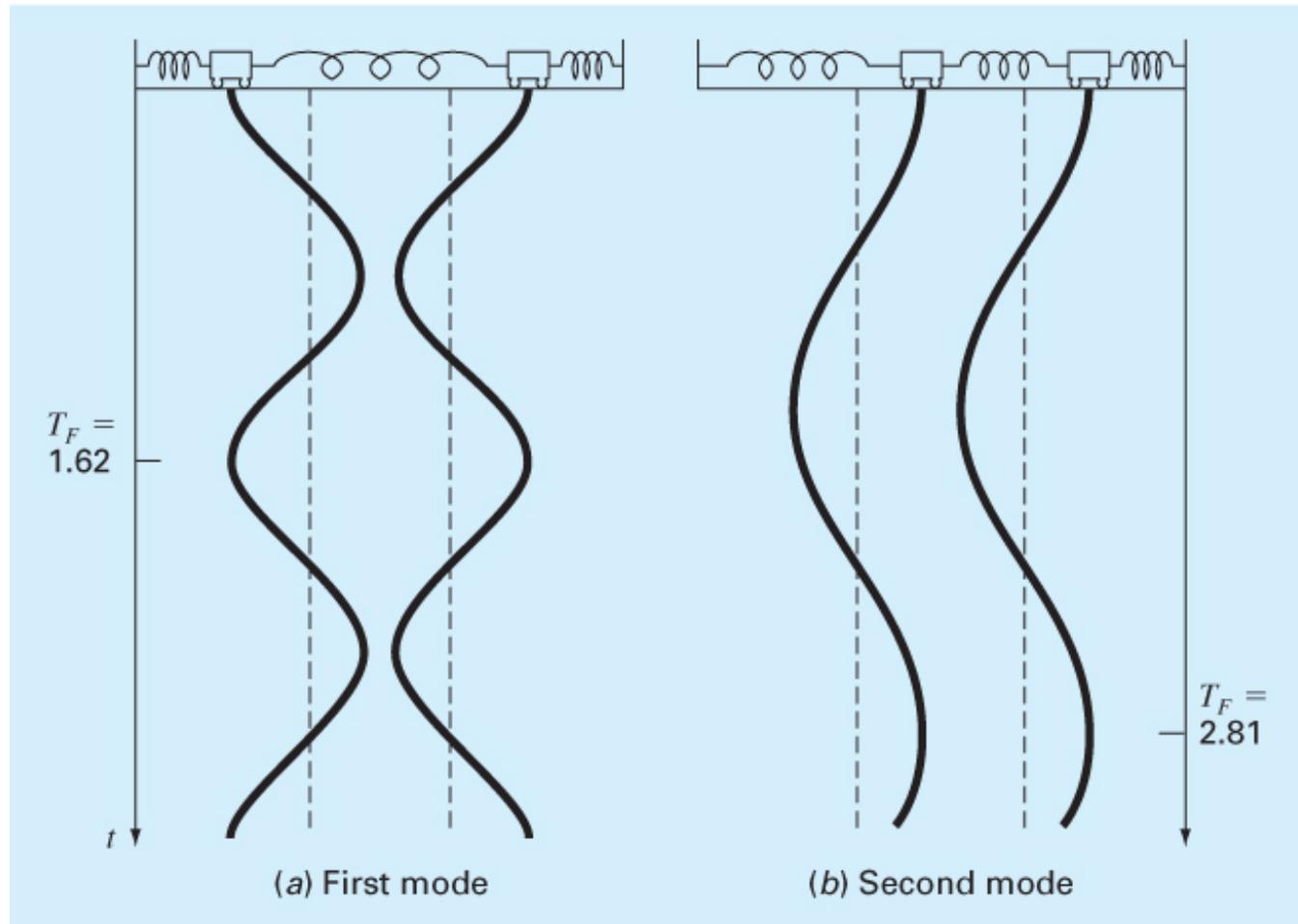


FIGURE 13.4

The principal modes of vibration of two equal masses connected by three identical springs between fixed walls.

The Power Method

- Iterative method to compute the *largest* eigenvalue and its associated eigenvector

$$[[A] - \lambda[I]]\{x\} = 0$$

$$[A]\{x\} = \lambda\{x\}$$

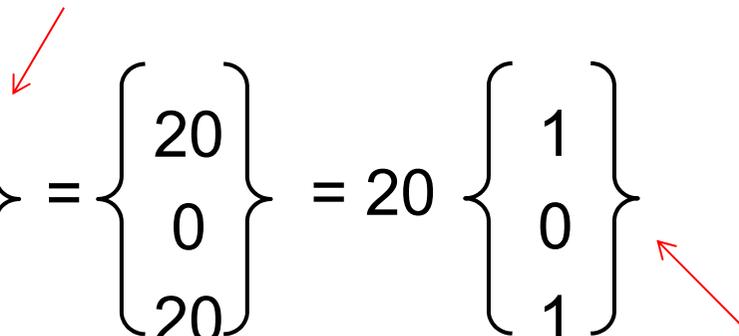
- Simple Algorithm:

```
function [eval, evect] = powereig(A,es,maxit)
n=length(A);
evect=ones(n,1);eval=1;iter=0;ea=100; %initialize
while(1)
    evalold=eval;           %save old eigenvalue value
    evect=A*evect;         %determine eigenvector as [A]*{x}
    eval=max(abs(evect));   %determine new eigenvalue
    evect=evect./eval;      %normalize eigenvector to eigenvalue
    iter=iter+1;
    if eval~=0, ea = abs((eval-evalold)/eval)*100; end
    if ea<=es | iter >= maxit,break,end
end
```

The Power Method: An Example (1/3)

Initial guesses of X's that
have all element equal to 1

- First iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 20 \end{Bmatrix} = 20 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$


Normalize the right-hand side vector
to make the large element equal to 1.

- Second iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 40 \\ -20 \\ 40 \end{Bmatrix} = 40 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{40 - 20}{40} \right| \times 100\% = 50\%$$

The Power Method: An Example (2/3)

- Third iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -80 \\ 60 \end{Bmatrix} = -80 \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{-80 - 40}{-80} \right| \times 100\% = 150\%$$

- Fourth iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix} = \begin{Bmatrix} -50 \\ 75 \\ -50 \end{Bmatrix} = 70 \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{70 - (-80)}{70} \right| \times 100\% = 214\%$$

The Power Method: An Example (3/3)

- Fifth iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix} = \begin{Bmatrix} -48.51714 \\ 68.51714 \\ -48.51714 \end{Bmatrix} = 68.51714 \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{68.51714 - 70}{70} \right| \times 100\% = 2.08\%$$

- The process can be continued to determine the largest eigenvalue (= 68.284) with the associated eigenvector $[-0.7071 \ 1 \ -0.7071]$

Note that the smallest eigenvalue and its associated eigenvector can be determined by applying the power method to the inverse of A

Determining Eigenvalues & Eigenvectors with MATLAB

eigenvector

```
>> A = [10 -5; -5 10]
```

```
A =  
10    -5  
-5    10
```

```
>> [v, lambda] = eig(A)
```

```
      v =  
  -0.7071  -0.7071  
  -0.7071   0.7071  
eigenvector eigenvector  
      lambda =  
eigenvalue  5      0  
             0     15  eigenvalue
```