Mathematical Modeling and Engineering Problem Solving

Berlin Chen
Department of Computer Science & Information Engineering
National Taiwan Normal University

Reference:

1. *Applied Numerical Methods with MATLAB for Engineers*, Chapter 1 & Teaching material

Chapter Objectives

- Provide a concrete idea of what numerical methods are and how they relate to engineering and scientific problem solving
  - Learning how mathematical models can be formulated on the basis of scientific principles to simulate the behavior of a simple physical system
  - Understanding how numerical methods afford a means to generalize solutions in a manner that can be implemented on a digital computer
  - Understanding the different types of conservation laws that lie beneath the models used in the various engineering disciplines and appreciating the difference between steady-state and dynamic solutions of these models
  - Learning about the different types of numerical methods we will cover in this book

A Simple Mathematical Model (1/2)

- A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms

- Models can be represented by a functional relationship between dependent variables, independent variables, parameters, and forcing functions

$$\begin{array}{c}\text{Dependent}\\ \text{variable}\end{array} = f\left(\begin{array}{c}\text{independent}\\ \text{variables}\end{array}, \text{parameters}, \begin{array}{c}\text{forcing}\\ \text{functions}\end{array}\right)$$

A Simple Mathematical Model (2/2)

- *Dependent variable* - a characteristic that usually reflects the behavior or state of the system
- *Independent variables* - dimensions, such as time and space, along which the system's behavior is being determined
- *Parameters* - constants reflective of the system's properties or composition
- *Forcing functions* - external influences acting upon the system

Modeling: Newton's Second Law of Motion

- **Statement:** The time rate of change of momentum of a body is equal to the resultant force acting on it

$$F = ma$$

$$\Rightarrow \quad a = \frac{F}{m}$$

forcing function

(net force acting on the object)

dependent variable

(acceleration)

Parameter

(mass of object)

  - No independent variable is involved
  - E.g., it can be used to determine the terminal velocity of a free-falling body near the earth's surface

More on Newton’s Second Law of Motion

- It describes a natural process or system in mathematical terms
- It represents an idealization and simplification of reality
  – Ignore negligible details of the natural process and focus on its essential manifestations
  – Exclude the effects of “relativity” that are of minimal importance when applied to object and forces that interact on or about the earth’s surface at velocities and on scales visible to humans
- It yields reproducible results and can be used for predictive purposes
  – Have generalization capabilities

Bungee-Jumping (1/2)

- For a body falling within the vicinity of the earth, the net force is composed of two opposing forces

$$F = F_D + F_U$$

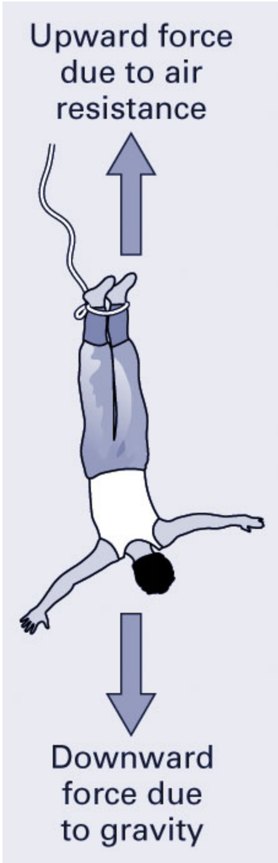
  - The downward pull of **gravity** F_D
    - The force due to gravity can be formulated as

$$F_D = mg$$

      - g is the acceleration due to gravity $(9.81\mathrm{m/s}^2)$
  - The upward force of **air resistance** F_U
    - A good approximation is to formulate it as

$$F_U = -c_d v^2$$

      - v is the velocity; c_d is the lumped drag coefficient, accounting for the properties of the falling object like shape or surface roughness
        - » The greater the fall velocity, the greater the upward force due to air resistance



Bungee-Jumping (2/2)

- The net force therefore is the difference between downward and upward force. We can a **differential equation** regarding the velocity of the object

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

- The exact solution of v **can not** be obtained using simple algebraic manipulation but rather using more advanced calculus techniques (when $v(t) = 0,\ t = 0$)

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

(an analytical or closed-form solution that can exactly satisfy the original differential equation)

Here t is independent variable,
$v(t)$ is dependent variable,
c_d and m are parameters,
and g is the forcing function.

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Example 1.1 (1/2)

- A bungee jumper with a mass of 68.1 kg leaps from a stationary hot air balloon (the drag coefficient is 0.25 kg/m)
  - Compute the velocity for the first 12 s of free fall
  - Determine the terminal velocity that will attained for an infinite long cord

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

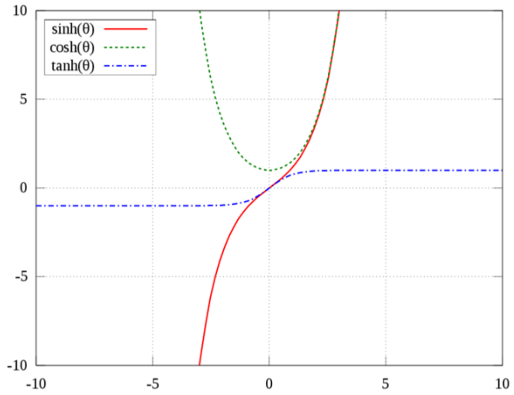
$$v(t) = \sqrt{\frac{9.8(68.1)}{0.25}} \tanh\left(\sqrt{\frac{9.8(0.25)}{68.1}} t\right) = 51.6938 \tanh(0.18977t)$$

$$\therefore v(12) = 50.6715$$

$$v(\infty) \approx 50.6938$$

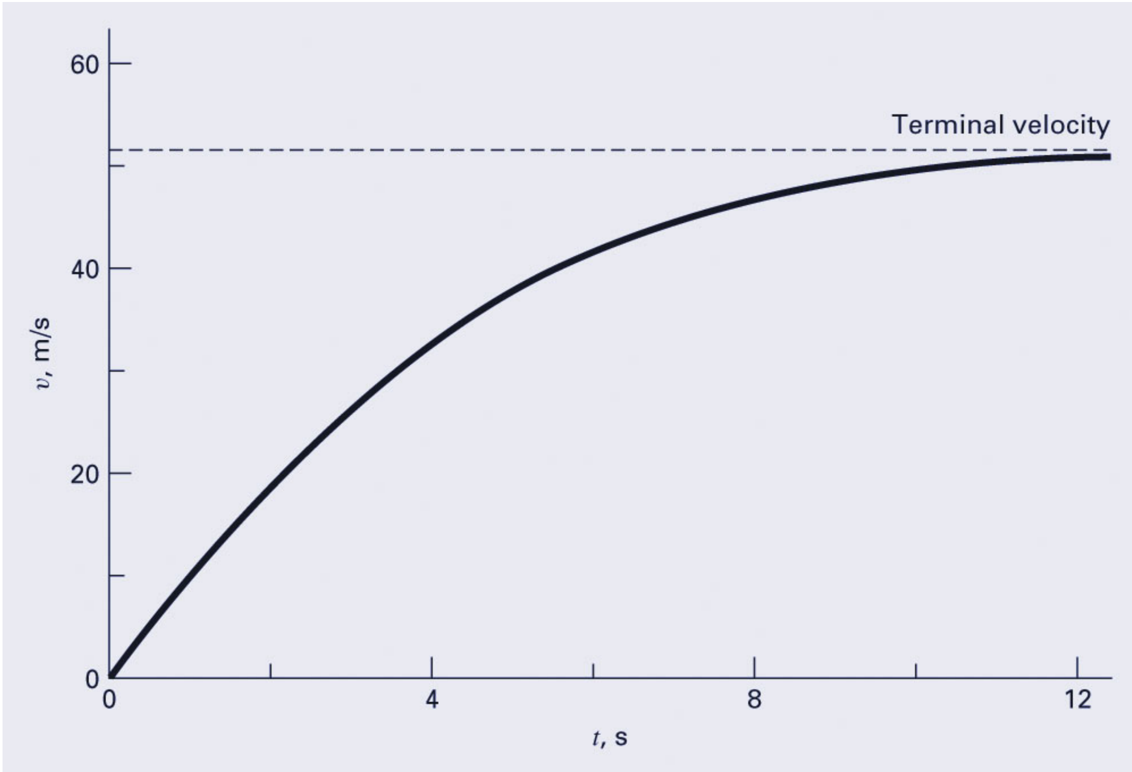
$$mg = c_d v^2$$

$$v = \sqrt{\frac{gm}{c_d}}$$



Example 1.1 (2/2)

- Using a computer (or a calculator), the model can be used to generate a graphical representation of the system



$$mg = c_d v^2$$

$$v = \sqrt{\frac{gm}{c_d}}$$

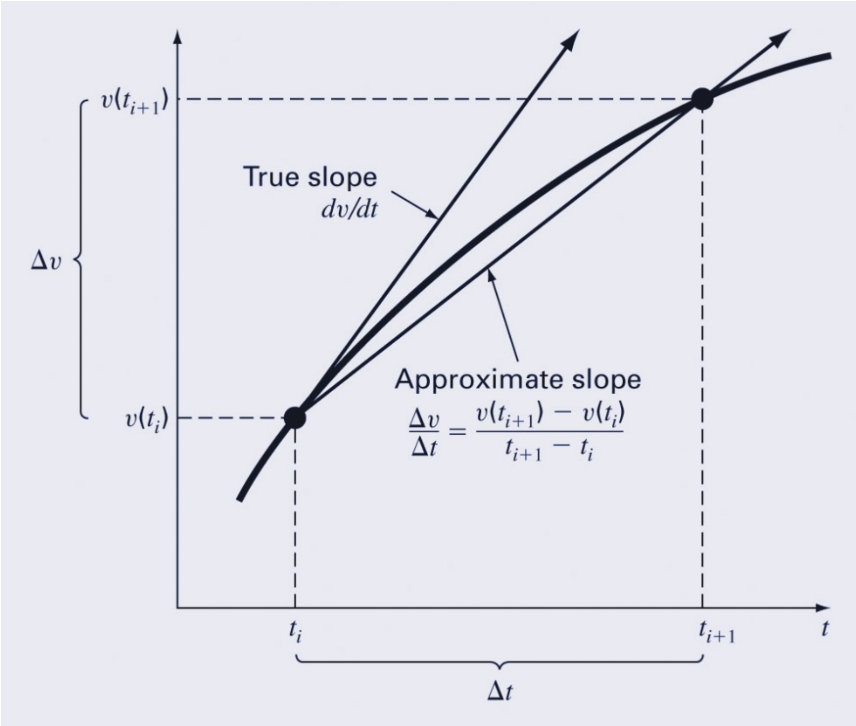
An example of Numerical Modeling (1/3)

- Numerical methods are those in which the mathematical problem is reformulated so it can be solved by arithmetic operations
  - E.g., the time rate of change of velocity mentioned earlier:

$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

(a finite-difference approximation of the derivate at time t_i)

$$\boxed{\text{Notice that } \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}}$$



An example of Numerical Modeling (2/3)

- Substituting the finite difference into the differential equation gives

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

$$\Rightarrow \quad \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v(t_i)^2$$

- Solve for

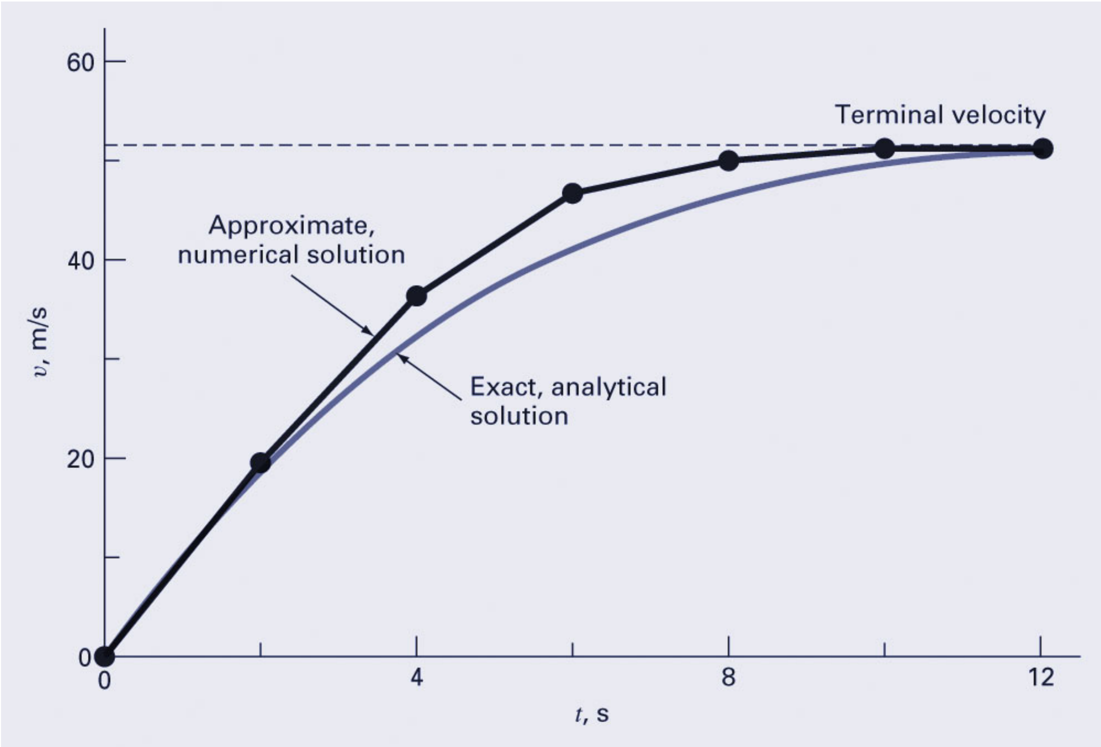
$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2 \right] (t_{i+1} - t_i)$$

new = old + slope $\times$ step

- This approach is formally called ***Euler's method***

An example of Numerical Modeling (3/3)

- Applying Euler's method in 2 s intervals yields:



- How do we improve the solution?
  - Smaller steps

Conservation laws (1/3)

- Conservation laws provide the foundation for many model functions
  - They boil down

$$\text{Change} = \text{increases} - \text{decreases}$$

  - Can be used to predict changes with respect to time by given it a special name “the *time-variable* (or *transient*)” computation
  - If no change occurs, the increases and decreases must be in balance

$$\text{Change} = 0 = \text{increases} - \text{decreases}$$

    - It is given a special name, the “*steady-state*” calculation

Conservation laws (2/3)

- Example: **Fluid Flow**

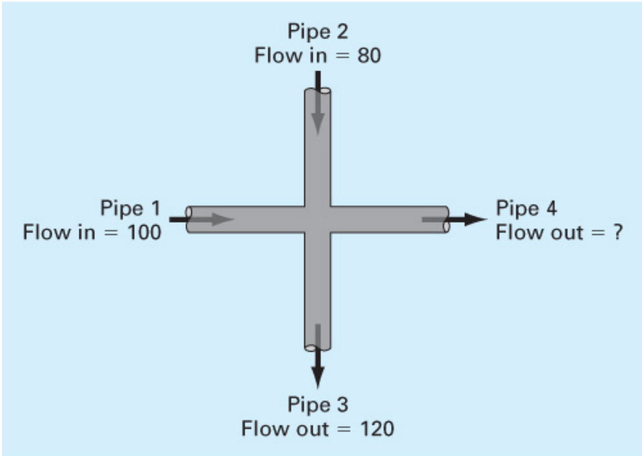


FIGURE 1.5
A flow balance for steady incompressible fluid flow at the junction of pipes.

  – For steady-state incompressible fluid flow in pipes

$$\text{Flow in} = \text{Flow out}$$

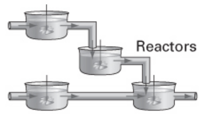
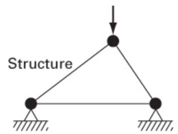
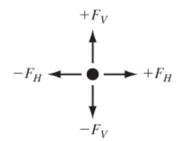
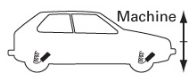
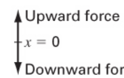
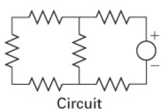
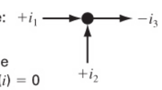
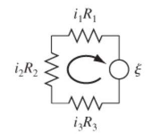
    • The flow out of the fourth pipe must be 60

Conservation laws (2/2)

- Different fields of engineering and science apply these laws to different paradigms within the field

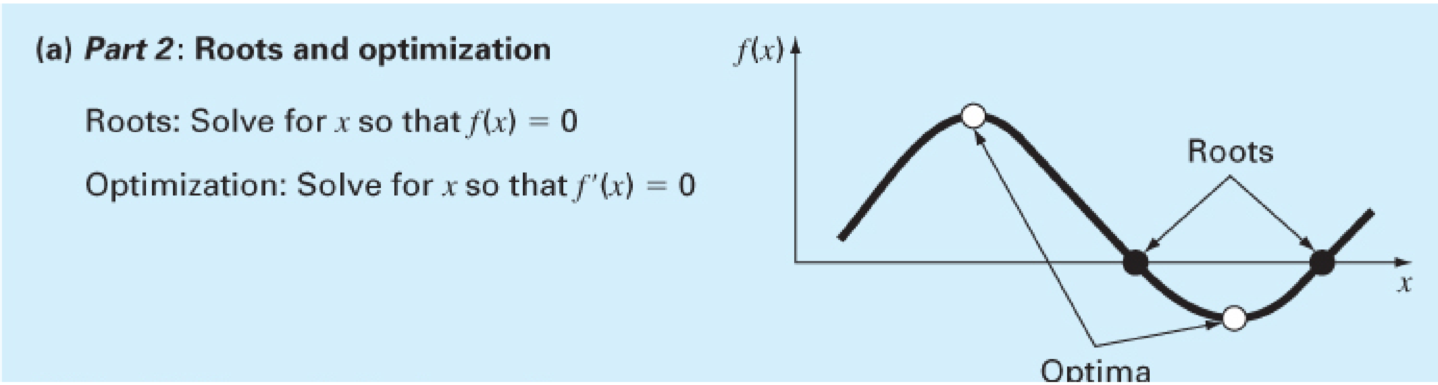
- Among these laws are
  - Conservation of mass
  - Conservation of momentum
  - Conservation of charge
  - Conservation of energy

TABLE 1.1 Devices and types of balances that are commonly used in the four major areas of engineering. For each case, the conservation law on which the balance is based is specified.

Field	Device	Organizing Principle	Mathematical Expression
Chemical engineering		Conservation of mass	Mass balance:  Over a unit of time period Δmass = inputs − outputs
Civil engineering		Conservation of momentum	Force balance:  At each node Σ horizontal forces (F_H) = 0 Σ vertical forces (F_V) = 0
Mechanical engineering		Conservation of momentum	Force balance:  $m\frac{d^2x}{dt^2}$ = downward force − upward force
Electrical engineering		Conservation of charge	Current balance:  For each node Σ current (i) = 0
		Conservation of energy	Voltage balance:  Around each loop Σ emf's − Σ voltage drops for resistors = 0 $\Sigma\,\xi - \Sigma\, iR = 0$

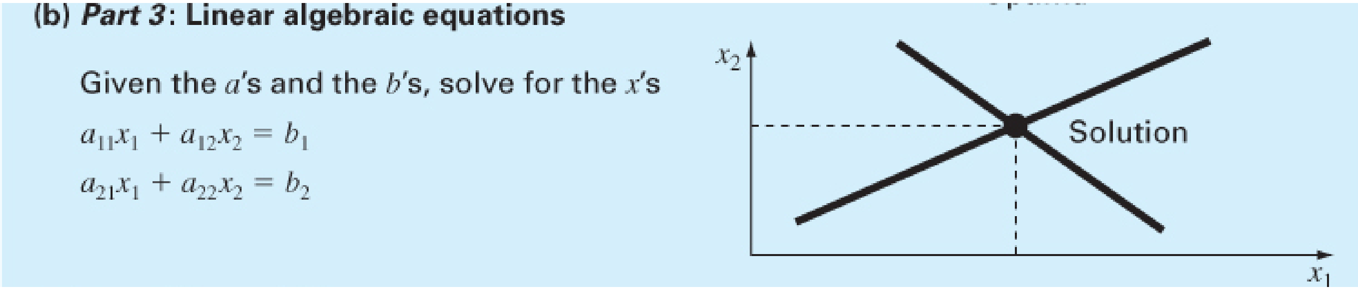
Summary of Numerical Methods (1/5)

- The book is divided into **five** categories of numerical methods
  - **Root Finding**: Search for the zero of a function
  - **Optimization**: Determine a value or values of an independent variable that correspond to a “best” or optimal value of a function



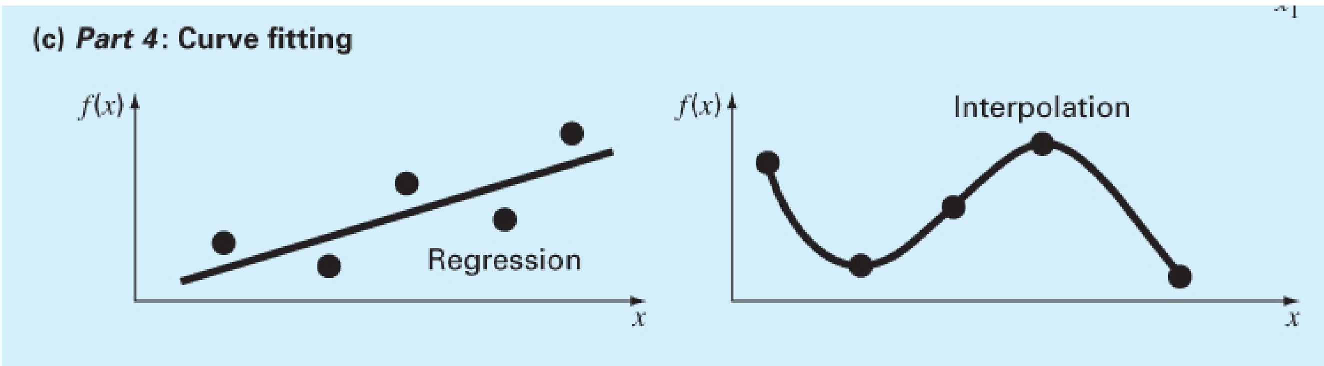
Summary of Numerical Methods (2/5)

– **Linear Algebraic Equations**: have to do with finding a set of values that simultaneously satisfy a set of linear algebraic equations



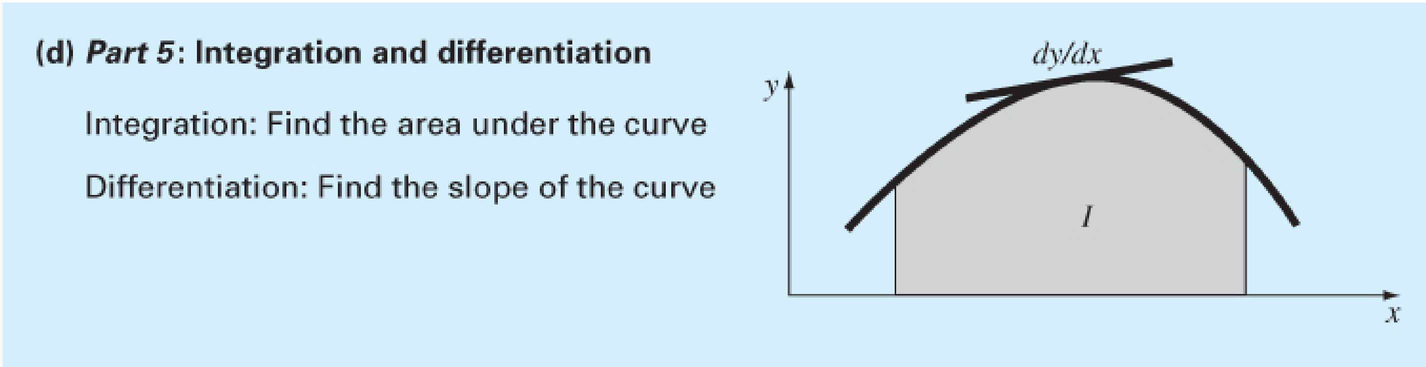
Summary of Numerical Methods (3/5)

– **Curve Fitting**
  - **Regression**: derive a single curve that represents the general trend of the data without necessarily matching any individual points
    – Is usually employed where there is significant degree of error associated with the data
  - **Interpolation**: determine intermediate values between relatively error-free data points
    – So as to generate tabulated information



Summary of Numerical Methods (4/5)

- **Integration**: determine the area under a curve
- **Differentiation**: determine a function's slope or its rate of change



Summary of Numerical Methods (5/5)

– **Differential Equation**: many physical laws are couched in terms of the rate of change of a quantity (rather than the magnitude of the quantity) which can be represented as differential equations
    - E.g., the acceleration of a falling body (bungee jumper)