Chapter 3

Euclidean Vector Spaces

Outline

- 3.1 Vectors in 2-Space, 3-Space, and n-Space
- 3.2 Norm, Dot Product, and Distance in R^n
- 3.3 Orthogonality
- 3.4 The Geometry of Linear Systems
- 3.5 Cross Product

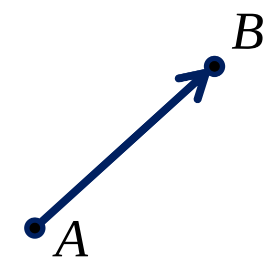
3.1

Vectors in 2-Space, 3-Space, and n-Space

Geometric Vectors

- In this text, vectors are denoted in bold face type such as **a**, **b**, **v**, and scalars are denoted in lowercase italic type such as *a*, *b*, *v*.
- A vector **v** has initial point *A* and terminal point *B*

$$\boldsymbol{v} = \overrightarrow{AB}$$



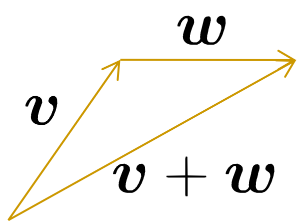
- Vectors with the same length and direction are said ***equivalent***.
- The vector whose initial and terminal points coincide (重疊) has length zero, and is called ***zero vector***, denoted by **0**.

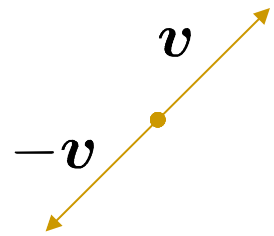
Definitions

- If **v** and **w** are any two vectors, then the sum **v** + **w** is the vector determined as follows:
  - Position the vector **w** so that its initial point coincides with the terminal point of **v**. The vector **v** + **w** is represented by the arrow from the initial point of **v** to the terminal point of **w**.
- If **v** and **w** are any two vectors, then the difference of **w** from **v** is defined by **v** – **w** = **v** + (-**w**).
- If **v** is a nonzero vector and *k* is nonzero real number (scalar), then the product *k***v** is defined to be the vector whose length is |*k*| times the length of **v** and whose direction is the same as that of **v** if $k > 0$ and opposite to that of **v** if $k < 0$. We define *k***v** = 0 if $k = 0$ or **v** = **0**.
- A vector of the form *k***v** is called a scalar multiple.

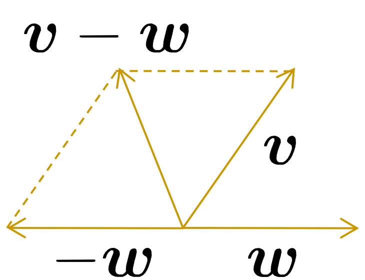
The negative of a vector **v**, denoted by –**v**, is the vector that has the same length as v but is oppositely directed.

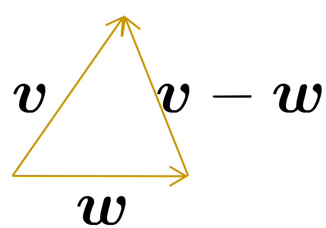
Examples (graphical illustration)

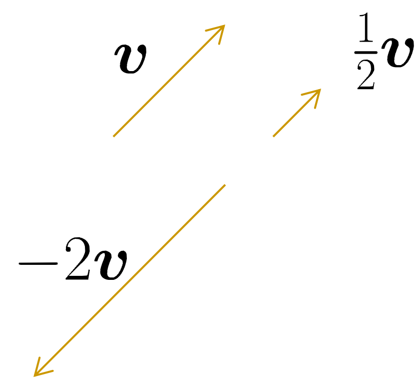




Position the initial point **of w** at the terminal point of **v** and draw a vector from the initial point of **v** to the terminal point of **w.**







Position **v** and **w** so their initial points coincide and draw a vector from the terminal point of **w** to the terminal point of **v.**

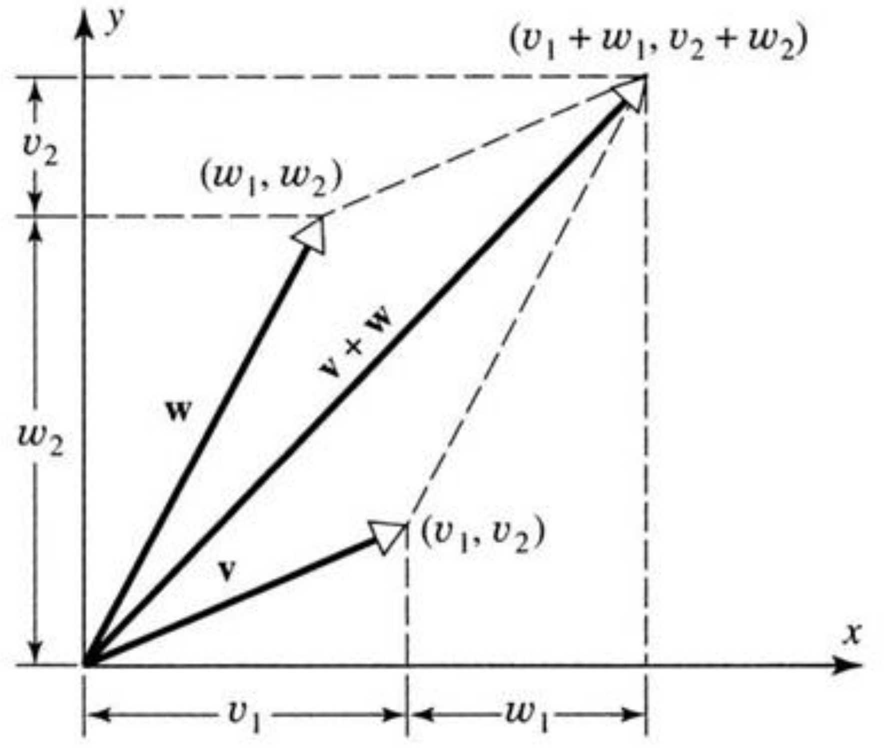
Vectors in Coordinate Systems

$$\boldsymbol{v} = (v_1, v_2)$$
$$\boldsymbol{w} = (w_1, w_2)$$

$$\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2)$$

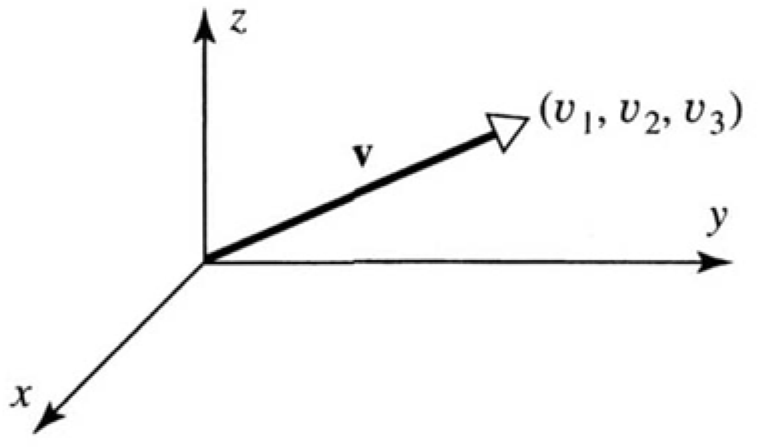
$$k\boldsymbol{v} = (kv_1, kv_2)$$

$$\boldsymbol{v} - \boldsymbol{w} = (v_1 - w_1, v_2 - w_2)$$



If a vector **v** in 2-space or 3-space is positioned with initial point at the original a rectangular coordinate system, then the vector is completely determined by the coordinates of its terminate point.
We call these coordinates the components of **v** relative to the coordinate system.

Vectors in 3-Space



$$\boldsymbol{v} = (v_1, v_2, v_3) \qquad \boldsymbol{w} = (w_1, w_2, w_3)$$

$$\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$$

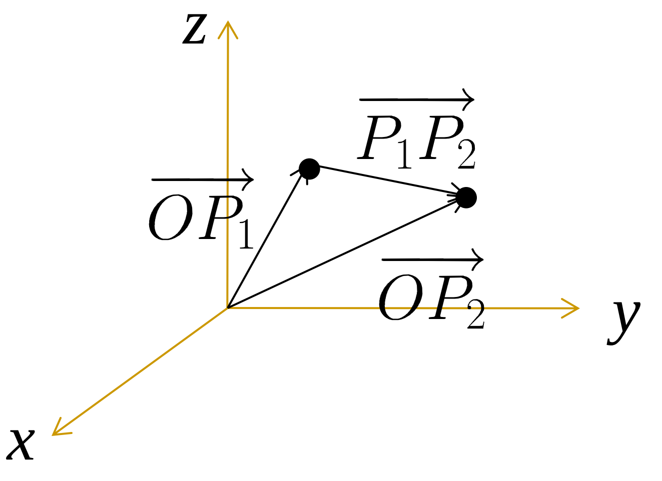
$$k\boldsymbol{v} = (kv_1, kv_2, kv_3)$$

$\boldsymbol{v}$ and $\boldsymbol{w}$ are equivalent if and only if $v_1=w_1$, $v_2=w_2$, $v_3=w_3$

Vectors

- If the vector $\overrightarrow{P_1P_2}$ has initial point P_1 (x_1, y_1, z_1) and terminal point P_2 (x_2, y_2, z_2), then

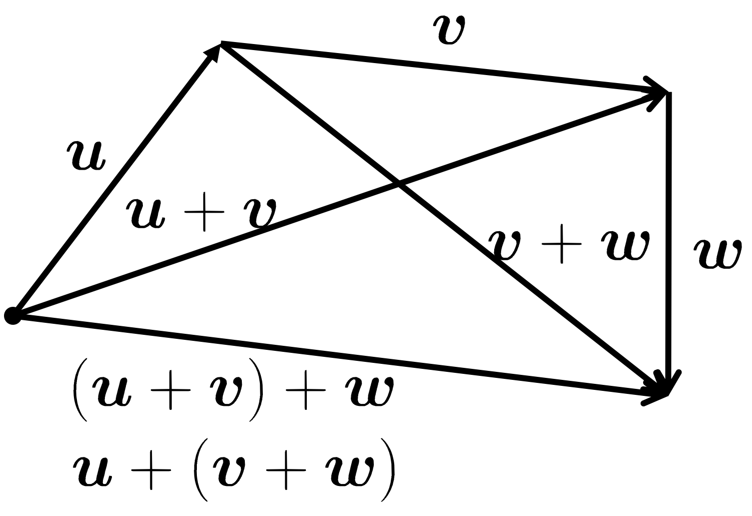
$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$



Theorem 3.1.1 (Properties of Vector Arithmetic)

- If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are vectors in R^n and k and l are scalars, then the following relationships hold.
  - $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  - $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
  - $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
  - $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
  - $k(l\mathbf{u}) = (kl)\mathbf{u}$
  - $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
  - $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
  - $1\mathbf{u} = \mathbf{u}$

Proof of part (b) (geometric)



Theorem and Definition

- Theorem 3.1.2: If **v** is a vector in R^n and k is a scalar, then:
  - $0\mathbf{v} = \mathbf{0}$
  - $k\mathbf{0} = \mathbf{0}$
  - $(-1)\mathbf{v} = -\mathbf{v}$
- If **w** is a vector in R^n, then **w** is said to be a ***linear combination*** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\boldsymbol{w} = k_1\boldsymbol{v}_1 + k_2\boldsymbol{v}_2 + \cdots + k_r\boldsymbol{v}_r$$

  - where $k_1, k_2, \ldots, k_r$ are scalars.

Alternative Notations for Vectors

- Comma-delimited form: $\boldsymbol{v} = (v_1, v_2, ..., v_n)$
- It can also written as a ***row-matrix*** form

$$\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

- Or a ***column-matrix*** form

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

3.2

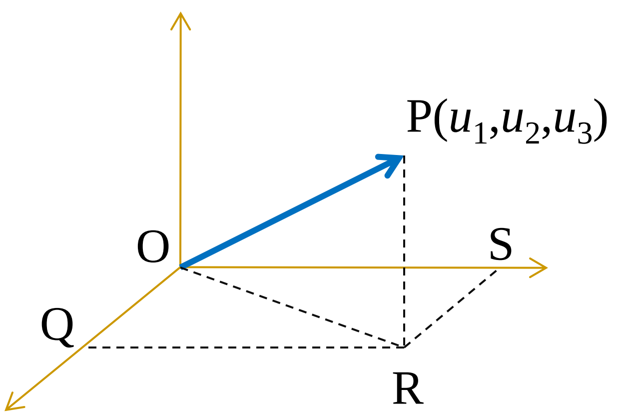
Norm, Dot Product, and Distance in R^n

Norm of a Vector

- The length of a vector **u** is often called the norm (範數) or magnitude of **u** and is denoted by ||**u**||.
- It follows from the **Theorem of Pythagoras** (畢達哥拉斯) that the norm of a vector **u** = (u_1,u_2,u_3) in 3-space is

$$\|\boldsymbol{u}\|^2 = (OR)^2 + (RP)^2$$
$$= (OQ)^2 + (QR)^2 + (RP)^2 = u_1^2 + u_2^2 + u_3^2$$

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$



Norm of a Vector

- If **v**=(v_1, v_2, …, v_n) is a vector in R^n, then the norm of **v** is denoted by ||**v**||, and is defined by

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- Example:
  - The norm of **v**=(-3,2,1) in R^3 is $\|\boldsymbol{v}\| = \sqrt{(-3)^2 + 2^2 + 1^2} = \sqrt{14}$
  - The norm of **v**=(2, -1, 3, -5) in R^4 is

$$\|\boldsymbol{v}\| = \sqrt{2^2 + (-1)^2 + 3^2 + (-5)^2} = \sqrt{39}$$

Theorem 3.2.1

- If **v** is a vector in R^n, and if k is any scalar, then:
  - $\|\mathbf{v}\| > 0$
  - $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$
  - $\|k\mathbf{v}\| = |k|\ \|\mathbf{v}\|$
- Proof of (c):
  - If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$, so

$$
\begin{aligned}
\|k\boldsymbol{v}\| &= \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2} \\
&= \sqrt{(k^2)(v_1^2 + v_2^2 + \cdots + v_n^2)} \\
&= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\
&= |k|\|\boldsymbol{v}\|
\end{aligned}
$$

Unit Vector

- A vector of norm 1 is called a unit vector. (單位向量)
- You can obtain a unit vector in a desired direction by choosing any nonzero vector **v** in that direction and multiplying **v** by the reciprocal of its length.

$$\boldsymbol{u} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

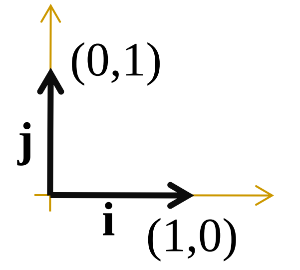
- The process is called *normalizing* **v**
- Example: **v** = (2,2,-1), $\|\boldsymbol{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$

$$\boldsymbol{u} = \tfrac{1}{3}(2, 2, -1) = (\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{-1}{3})$$

  - You can verify that $\|\boldsymbol{u}\| = 1$

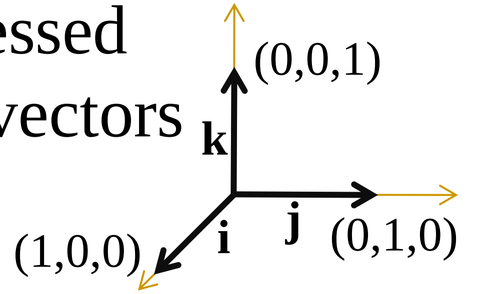
Standard Unit Vectors

- When a **rectangular coordinate system** is introduced in R^2 or R^3, the unit vectors in the positive directions of the coordinates axes are called ***standard unit vectors***.
- In R^2, $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$
- In R^3, $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$



- Every vector $\mathbf{v}=(v_1,v_2)$ in R^2 can be expressed as a linear combination of standard unit vectors

$$\boldsymbol{v} = (v_1, v_2) = v_1(1,0) + v_2(0,1) = v_1\boldsymbol{i} + v_2\boldsymbol{j}$$



Standard Unit Vectors

- We can generalize these formulas to R^n by defining standard unit vectors in R^n to be

$$\boldsymbol{e}_1 = (1, 0, 0, \ldots, 0) \quad \boldsymbol{e}_2 = (0, 1, 0, \ldots, 0) \quad \ldots \quad \boldsymbol{e}_n = (0, 0, 0, \ldots, 1)$$

- Every vector $\mathbf{v}=(v_1,v_2,\ldots,v_n)$ in R^n can be expressed as

$$\boldsymbol{v} = (v_1, v_2, \ldots, v_n) = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + \cdots + v_n\boldsymbol{e}_n$$

- Example: $(2,-3,4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
- $(7,3,-4,5) = 7\mathbf{e}_1 + 3\mathbf{e}_2 - 4\mathbf{e}_3 + 5\mathbf{e}_4$

Distance

- The distance between two points is the norm of the vector.
- If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are two points in 3-space, then the distance d between them is the norm of the vector $\overrightarrow{P_1P_2}$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Euclidean distance (歐幾里德距離, 歐式距離)
- If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n, then the distance $d(\mathbf{u},\mathbf{v})$ is defined as

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Definitions

- Let **u** and **v** be two nonzero vectors in 2-space or 3-space, and assume these vectors have been positioned so their initial points coincided. By the angle between **u** and **v**, we shall mean the angle θ determined by **u** and **v** that satisfies $0 \le \theta \le \pi$.
- If **u** and **v** are vectors in 2-space or 3-space and θ is the angle between **u** and **v**, then the dot product (點積) or Euclidean inner product (內積) $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases}$$

Dot Product

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}$$

- If the vectors **u** and **v** are nonzero and θ is the angle between them, then
  - θ is acute (銳角) if and only if $\mathbf{u}\cdot\mathbf{v} > 0$
  - θ is obtuse (鈍角) if and only if $\mathbf{u}\cdot\mathbf{v} < 0$
  - $\theta = \pi/2$ (直角) if and only if $\mathbf{u}\cdot\mathbf{v} = 0$

Example

- If the angle between the vectors **u** = (0,0,1) and **v** = (0,2,2) is 45°, then

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = \sqrt{0+0+1}\sqrt{0+4+4}\cdot\left(\frac{1}{\sqrt{2}}\right) = 2$$

$$\mathbf{u}\cdot\mathbf{v} = (u_1, u_2, u_3)\cdot(v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3 = 2$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2}{\sqrt{0+0+1}\sqrt{0+4+4}} = \frac{1}{\sqrt{2}}$$

Component Form of Dot Product

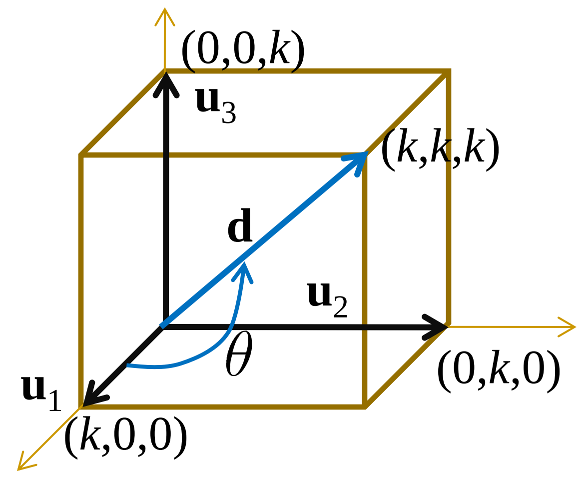
Example

- Find the angle between a diagonal of a cube and one of its edges

$$\boldsymbol{d} = (k, k, k) = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$$

$$\cos\theta = \frac{\boldsymbol{u}_1 \cdot \boldsymbol{d}}{\|\boldsymbol{u}_1\|\|\boldsymbol{d}\|} = \frac{k^2}{(k)(\sqrt{3k^2})} = \frac{1}{\sqrt{3}}$$

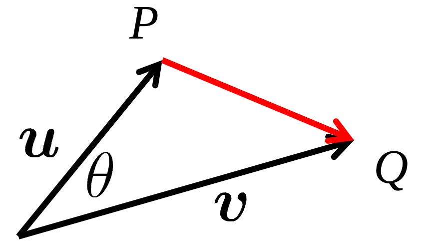
$$\theta = \cos^{-1}(\tfrac{1}{\sqrt{3}}) \approx 54.74^\circ$$



Component Form of Dot Product

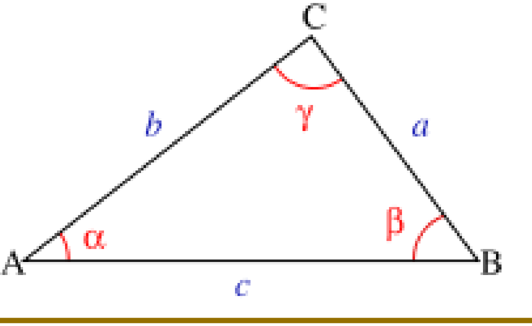
- Let $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{v}=(v_1,v_2,v_3)$ be two nonzero vectors.
- According to the ***law of cosine*** (餘弦定理)

$$\|\overrightarrow{PQ}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{u}\|\|\boldsymbol{v}\|\cos\theta$$



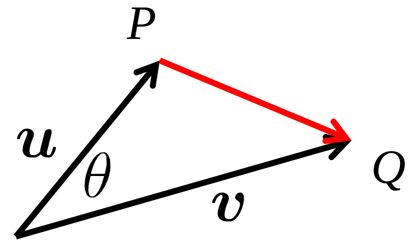
law of cosine

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$



Component Form of Dot Product

$$\|\overrightarrow{PQ}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\cos\theta$$



- $\overrightarrow{PQ} = \boldsymbol{v} - \boldsymbol{u}$
- $\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|\cos\theta = \frac{1}{2}(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{v} - \boldsymbol{u}\|^2)$
- $\boldsymbol{u}\cdot\boldsymbol{v} = \frac{1}{2}(\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{v} - \boldsymbol{u}\|^2)$
- $\boldsymbol{u}\cdot\boldsymbol{v} = u_1v_1 + u_2v_2 + u_3v_3$

$$\|\boldsymbol{u}\|^2 = u_1^2 + u_2^2 + u_3^2$$
$$\|\boldsymbol{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\|\boldsymbol{v} - \boldsymbol{u}\|^2 = (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2$$

Definition

- If **u**=(u_1,u_2,…,u_n) and **v**=(v_1,v_2,…,v_n) are vectors in R^n, then the **dot product** (點積) (also called the **Euclidean inner product** (內積)) of **u** and **v** is denoted by **u**・**v** and is defined by

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Example: **u**=(-1,3,5,7) and **v**=(-3,-4,1,0)
  - **u**・**v** = (-1)(-3) + (3)(-4) + (5)(1) + (7)(0) = -4

Theorems

$$\mathbf{u}\cdot\mathbf{v}=\begin{cases}\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta & \text{if } \mathbf{u}\neq\mathbf{0} \text{ and } \mathbf{v}\neq\mathbf{0}\\ 0 & \text{if } \mathbf{u}=\mathbf{0} \text{ or } \mathbf{v}=\mathbf{0}\end{cases}$$

- The special case **u** = **v**, we obtain the relationship

$$\boldsymbol{v}\cdot\boldsymbol{v}=v_1^2+v_2^2+\cdots+v_n^2=\|\boldsymbol{v}\|^2$$

$$\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$$

- Theorem 3.2.2
  - If **u**, **v** and **w** are vectors in 2- or 3-space, and *k* is a scalar, then
    - $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ [symmetry property]
    - $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ [distributive property]
    - $k(\mathbf{u}\cdot\mathbf{v})=(k\mathbf{u})\cdot\mathbf{v}=\mathbf{u}\cdot(k\mathbf{v})$ [homogeneity property]
    - $\mathbf{v}\cdot\mathbf{v}\geqq 0$ and $\mathbf{v}\cdot\mathbf{v}=0$ if $\mathbf{v}=0$ [positivity property]

Proof of Theorem 3.2.2

$k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

- Let $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{v}=(v_1,v_2,v_3)$

$$
\begin{aligned}
k(\boldsymbol{u} \cdot \boldsymbol{v}) &= k(u_1v_1 + u_2v_2 + u_3v_3) \\
&= (ku_1)v_1 + (ku_2)v_2 + (ku_3)v_3 \\
&= (k\boldsymbol{u}) \cdot \boldsymbol{v}
\end{aligned}
$$

Theorem 3.2.3

- If **u**, **v**, and **w** are vectors in R^n, and if k is a scalar, then
  - $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$
  - $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
  - $\mathbf{u} \cdot (\mathbf{v}-\mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$
  - $(\mathbf{u}-\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$
  - $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$
- Proof(b)

$$(\boldsymbol{u}+\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{w} \cdot (\boldsymbol{u}+\boldsymbol{v}) \quad \text{[by symmetry]}$$
$$= \boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w} \cdot \boldsymbol{v} \quad \text{[by distributivity]}$$
$$= \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w} \quad \text{[by symmetry]}$$

Example

- Calculating with dot products
  - $(\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v})$
    $= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v})$
    $= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v})$
    $= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2$

Cauchy-Schwarz Inequality

- With the formula

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \qquad \theta = \cos^{-1}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|}\right)$$

- The inverse cosine is not defined unless its argument satisfies the inequalities

$$-1 \le \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\,\|\boldsymbol{v}\|} \le 1$$

- Fortunately, these inequalities do hold for all nonzero vectors in R^n as a result of **Cauchy-Schwarz inequality**

Theorem 3.2.4 Cauchy-Schwarz Inequality

- If **u** = $(u_1, u_2, \ldots, u_n)$ and **v**=$(v_1, v_2, \ldots, v_n)$ are vectors in R^n, then $|\mathbf{u} \cdot \mathbf{v}| \leqq \|\mathbf{u}\| \, \|\mathbf{v}\|$
or in terms of components

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n|$$
$$\leq (u_1^2 + u_2^2 + \cdots + u_n^2)^{1/2}(v_1^2 + v_2^2 + \cdots + v_n^2)^{1/2}$$

We will omit the proof of this theorem because later in the text we will prove a more general version of which this will be a special case.

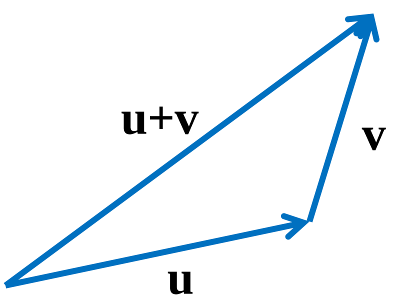
- To show $-1 \leq \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \leq 1$

$$-1 \leq \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \leq 1 \Rightarrow \frac{|\boldsymbol{u} \cdot \boldsymbol{v}|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \leq 1 \Rightarrow \left|\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right| \leq 1$$

- Cauchy-Schwarz Inequality:
  If **u** = $(u_1, u_2, \ldots, u_n)$ and **v**$=(v_1, v_2, \ldots, v_n)$ are vectors in R^n,
  then $|\mathbf{u} \cdot \mathbf{v}| \leqq \|\mathbf{u}\| \, \|\mathbf{v}\|$

Geometry in R^n

- The sum of the lengths of two side of a triangle is at least as large as the third
- The shortest distance between two points is a straight line
- Theorem 3.2.5
  - If **u**, **v**, and **w** are vectors in R^n, and k is any scalar, then
  - $\|\mathbf{u}+\mathbf{v}\| \leqq \|\mathbf{u}\| + \|\mathbf{v}\|$
  - $d(\mathbf{u},\mathbf{v}) \leqq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$



Proof of Theorem 3.2.5

- Proof (a)

$$\begin{aligned}
\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) \\
&= (\boldsymbol{u}\cdot\boldsymbol{u}) + 2(\boldsymbol{u}\cdot\boldsymbol{v}) + (\boldsymbol{v}\cdot\boldsymbol{v}) \\
&= \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v}) + \|\boldsymbol{v}\|^2 \\
&\le \|\boldsymbol{u}\|^2 + 2|\boldsymbol{u}\cdot\boldsymbol{v}| + \|\boldsymbol{v}\|^2 && \longleftarrow \text{Property of absolute value} \\
&\le \|\boldsymbol{u}\|^2 + 2\|\boldsymbol{u}\|\|\boldsymbol{v}\| + \|\boldsymbol{v}\|^2 && \longleftarrow \text{Cauchy-Schwarz inequality} \\
&= (\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)^2
\end{aligned}$$

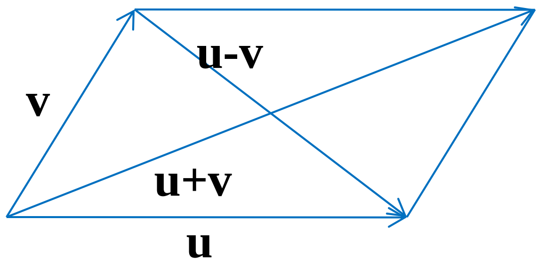
- Proof (b)

$$\begin{aligned}
d(\boldsymbol{u},\boldsymbol{v}) &= \|\boldsymbol{u}-\boldsymbol{v}\| \\
&= \|(\boldsymbol{u}-\boldsymbol{w}) + (\boldsymbol{w}-\boldsymbol{v})\| \\
&\le \|\boldsymbol{u}-\boldsymbol{w}\| + \|\boldsymbol{w}-\boldsymbol{v}\| && \text{based on (a)} \\
&= d(\boldsymbol{u},\boldsymbol{w}) + d(\boldsymbol{w},\boldsymbol{v})
\end{aligned}$$

Theorem 3.2.6 Parallelogram Equation for Vectors

- If u and v are vectors in R^n, then
  $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$
- Proof:

$$
\begin{aligned}
&\|\boldsymbol{u}+\boldsymbol{v}\|^2+\|\boldsymbol{u}-\boldsymbol{v}\|^2 \\
&=(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v})+(\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v}) \\
&=2(\boldsymbol{u}\cdot\boldsymbol{u})+2(\boldsymbol{v}\cdot\boldsymbol{v}) \\
&=2(\|\boldsymbol{u}\|^2+\|\boldsymbol{v}\|^2)
\end{aligned}
$$



Theorem 3.2.7

- If **u** and **v** are vectors in R^n with the Euclidean inner product, then $\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u} + \boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u} - \boldsymbol{v}\|^2$
- Proof:

$$\|\boldsymbol{u} + \boldsymbol{v}\|^2 = (\boldsymbol{u} + \boldsymbol{v}) \cdot (\boldsymbol{u} + \boldsymbol{v}) = \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u} \cdot \boldsymbol{v}) + \|\boldsymbol{v}\|^2$$

$$\|\boldsymbol{u} - \boldsymbol{v}\|^2 = (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) = \|\boldsymbol{u}\|^2 - 2(\boldsymbol{u} \cdot \boldsymbol{v}) + \|\boldsymbol{v}\|^2$$

Dot Products as Matrix Multiplication

- View **u** and **v** as column matrices

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = \boldsymbol{v}^T \boldsymbol{u}$$

- Example:

$$\boldsymbol{u} = (1, -3, 5) \qquad \boldsymbol{v} = (5, 4, 0)$$

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1, -3, 5) \cdot (5, 4, 0) = (1)(5) + (-3)(4) + (5)(0) = -7$$

$$\boldsymbol{u}^T \boldsymbol{v} = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = -7 \qquad \boldsymbol{v}^T \boldsymbol{u} = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = -7$$

Dot Products as Matrix Multiplication

- If A is an $n \times n$ matrix and **u** and **v** are $n \times 1$ matrices

$$A\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v}^T(A\boldsymbol{u}) = (\boldsymbol{v}^T A)\boldsymbol{u} = (A^T\boldsymbol{v})^T\boldsymbol{u} = \boldsymbol{u} \cdot A^T\boldsymbol{v}$$

$$\boldsymbol{u} \cdot A\boldsymbol{v} = (A\boldsymbol{v})^T\boldsymbol{u} = (\boldsymbol{v}^T A^T)\boldsymbol{u} = \boldsymbol{v}^T(A^T\boldsymbol{u}) = A^T\boldsymbol{u} \cdot \boldsymbol{v}$$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

You can check $A\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot A^T\boldsymbol{v}$

Dot Product View of Matrix Multiplication

- If $A=[a_{ij}]$ is a $m \times r$ matrix, and $B=[b_{ij}]$ is an $r \times n$ matrix, then the ijth entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

which is the dot product of the ith row vector of A

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{ir} \end{bmatrix}$$

and the jth column vector of B

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}$$

Dot Product View of Matrix Multiplication

- If the row vectors of A are $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_m$ and the column vectors of B are $\mathbf{c}_1$, $\mathbf{c}_2$, …, $\mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{c}_1 & \boldsymbol{r}_1 \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{r}_1 \cdot \boldsymbol{c}_n \\ \boldsymbol{r}_2 \cdot \boldsymbol{c}_1 & \boldsymbol{r}_2 \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{r}_2 \cdot \boldsymbol{c}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{r}_m \cdot \boldsymbol{c}_1 & \boldsymbol{r}_m \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{r}_m \cdot \boldsymbol{c}_n \end{bmatrix}$$

3.3
Orthogonality

Orthogonal Vectors

- Recall that $\theta = \cos^{-1}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$
- It follows that $\theta = \frac{\pi}{2}$ if and only if $\mathbf{u} \cdot \mathbf{v} = 0$
- Definition: Two nonzero vectors **u** and **v** in R^n are said to be ***orthogonal*** [正交] (or ***perpendicular*** [垂直]) if $\mathbf{u} \cdot \mathbf{v} = 0$.
- The zero vector in R^n is orthogonal to every vector in R^n.
- A nonempty set of vectors in R^n is called an ***orthogonal set*** if all pairs of distinct vectors in the set are orthogonal.
- An orthogonal set of unit vectors is called an ***orthonormal set***.

Example

- Show that **u**=(-2,3,1,4) and **v**=(1,2,0,-1) are orthogonal

$$\boldsymbol{u} \cdot \boldsymbol{v} = (-2)(1) + (3)(2) + (1)(0) + (4)(-1) = 0$$

- Show that the set S={**i**,**j**,**k**} of standard unit vectors is an orthogonal set in R^3
  - We must show $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$

$$\boldsymbol{i} \cdot \boldsymbol{j} = (1,0,0) \cdot (0,1,0) = 0$$
$$\boldsymbol{i} \cdot \boldsymbol{k} = (1,0,0) \cdot (0,0,1) = 0$$
$$\boldsymbol{j} \cdot \boldsymbol{k} = (0,1,0) \cdot (0,0,1) = 0$$

Normal

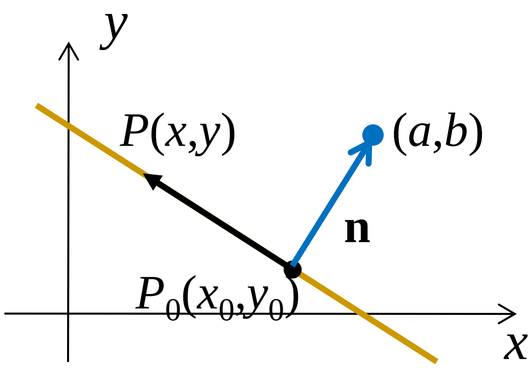
- One way of specifying slope and inclination is the use a nonzero vector **n**, called ***normal*** (法向量) that is orthogonal to the line or plane.

$$\boldsymbol{n} \cdot \overrightarrow{P_0P} = 0$$

$$a(x - x_0) + b(y - y_0) = 0$$

The line through the point (x_0,y_0) has normal $\mathbf{n}=(a,b)$

Example: the equation $6(x\text{-}3) + (y+7) = 0$ represents the line through (3,-7) with normal $\mathbf{n}=(6,1)$



Theorem 3.3.1

- If a and b are constants that are not both zero, then an equation of the form $ax+by+c = 0$ represents a line in R^2 with normal $\mathbf{n}=(a,b)$
- If a, b, and c are constants that are not all zero, then an equation of the form $ax+by+cz+d = 0$ represents a plane in R^3 with normal $\mathbf{n}=(a,b,c)$

Example

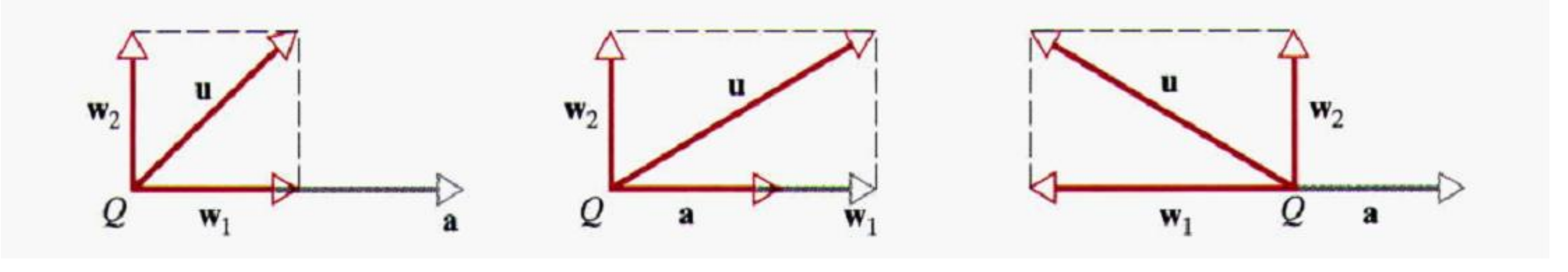
- The equation $ax+by=0$ represents a line through the origin in R^2. Show that the vector $\mathbf{n}=(a,b)$ is orthogonal to the line, that is, orthogonal to every vector along the line.
- Solution:
  - Rewrite the equation as

$$(a,b)\cdot(x,y)=0$$
$$\boldsymbol{n}\cdot(x,y)=0$$

Therefore, the vector **n** is orthogonal to every vector (x,y) on the line.

An Orthogonal Projection

- To "decompose" a vector **u** into a sum of two terms, one *parallel* to a specified nonzero vector **a** and the other *perpendicular* to **a**.
- We have $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ and $\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{w}_1 + (\mathbf{u} - \mathbf{w}_1) = \mathbf{u}$
- The vector $\mathbf{w}_1$ is called the orthogonal projection (正交投影) of **u** on **a** or sometimes the vector component (分向量) of **u** along **a**, and denoted by $\text{proj}_{\mathbf{a}}\mathbf{u}$
- The vector $\mathbf{w}_2$ is called the vector component of **u** orthogonal to **a**, and denoted by $\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$



Theorem 3.3.2 Projection Theorem

- If **u** and **a** are vectors in R^n, and if $\mathbf{a}\neq\mathbf{0}$, then **u** can be expressed in exactly one way in the form $\mathbf{u}=\mathbf{w}_1+\mathbf{w}_2$, where $\mathbf{w}_1$ is a scalar multiple of **a** and $\mathbf{w}_2$ is orthogonal to **a**.
- Proof:
  - Since $\mathbf{w}_1$ is to be a scalar multiple of **a**, it has the form: $\mathbf{w}_1 = k\mathbf{a}$
  - Our goal is to find a value of k and a vector $\mathbf{w}_2$ that is orthogonal to **a** such that $\mathbf{u}=\mathbf{w}_1+\mathbf{w}_2$.
  - Rewrite $\mathbf{u}=\mathbf{w}_1+\mathbf{w}_2=k\mathbf{a}+\mathbf{w}_2$, and then applying Theorems 3.2.2 and 3.2.3 to obtain $\mathbf{u}\cdot\mathbf{a}=(k\mathbf{a}+\mathbf{w}_2)\cdot\mathbf{a}=k\|\mathbf{a}\|^2+(\mathbf{w}_2\cdot\mathbf{a})$
  - Since $\mathbf{w}_2$ is orthogonal to **a**, $\mathbf{u}\cdot\mathbf{a} = k\|\mathbf{a}\|^2$, from which we obtain $k=\dfrac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}$
  - Therefore, we can get

$$\boldsymbol{w}_2=\boldsymbol{u}-\boldsymbol{w}_1=\boldsymbol{u}-k\boldsymbol{a}=\boldsymbol{u}-\frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}$$

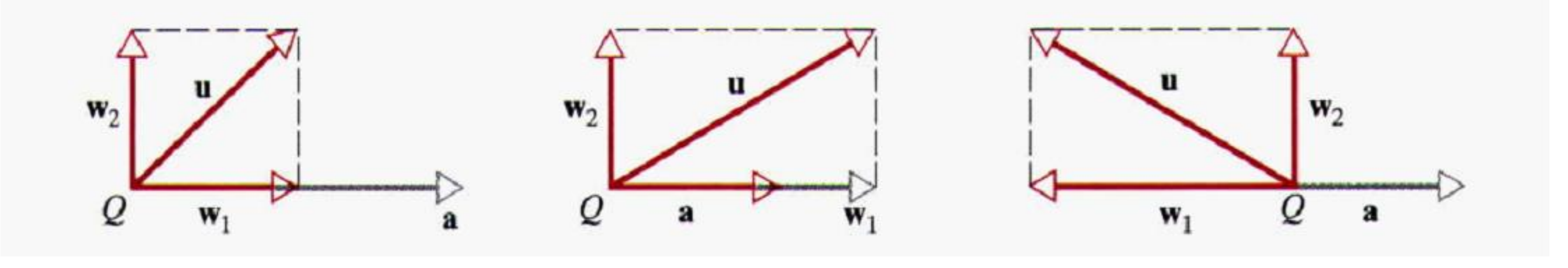
Projection Theorem

$$\mathbf{w}_1 = \text{proj}_{\mathbf{a}} \mathbf{u}$$
$$\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}} \mathbf{u}$$

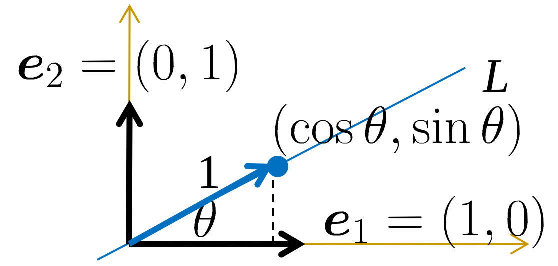
- The vector $\mathbf{w}_1$ is called ***the orthogonal projection of* u *on* a**, or ***the vector component of* u *along* a.**
- The vector $\mathbf{w}_2$ is called ***the vector component of* u *orthogonal to* a**.

$$\text{proj}_{\mathbf{a}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} \qquad \text{(vector component of } \mathbf{u} \text{ along } \mathbf{a}\text{)}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}} \mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a} \qquad \text{(vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{a}\text{)}$$



Example



- Find the orthogonal projections of the vectors $\mathbf{e}_1$=(1,0) and $\mathbf{e}_2$=(0,1) on the line L that makes an angle θ with the positive x-axis in R2.
- Solution:
  - $\boldsymbol{a} = (\cos\theta, \sin\theta)$ is a unit vector along L.
  - Find orthogonal projection of $\mathbf{e}_1$ along $\mathbf{a}$.

$$\|\boldsymbol{a}\| = \sqrt{\sin\theta^2 + \cos\theta^2} = 1 \qquad \boldsymbol{e}_1 \cdot \boldsymbol{a} = (1,0)\cdot(\cos\theta, \sin\theta) = \cos\theta$$

$$proj_{\boldsymbol{a}}\boldsymbol{e}_1 = \frac{\boldsymbol{e}_1 \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = (\cos\theta)(\cos\theta, \sin\theta) = (\cos\theta^2, \sin\theta\cos\theta)$$

$$\boldsymbol{e}_2 \cdot \boldsymbol{a} = (0,1)\cdot(\cos\theta, \sin\theta) = \sin\theta$$

$$proj_{\boldsymbol{a}}\boldsymbol{e}_2 = \frac{\boldsymbol{e}_2 \cdot \boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = (\sin\theta)(\cos\theta, \sin\theta) = (\sin\theta\cos\theta, \sin\theta^2)$$

Example

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

Let $u = (2,-1,3)$ and $a = (4,-1,2)$. Find the vector component of u along a and the vector component of u orthogonal to a.

- Solution:

$$u \cdot a = (2)(4) + (-1)(-1) + (3)(2) = 15$$

$$\|a\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

Thus, the vector component of u along a is

$$proj_a\, u = \frac{u \cdot a}{\|a\|^2} a = \tfrac{15}{21}(4,-1,2) = (\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7})$$

and the vector component of u orthogonal to a is

$$u - proj_a\, u = (2,-1,3) - (\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}) = (-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{11}{7})$$

Verify that the vector $u - proj_a\, u$ and a are perpendicular by showing that their dot product is zero.

Length of Orthogonal Projection

$$\|proj_{\boldsymbol{a}}\boldsymbol{u}\| = \left\|\frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\boldsymbol{a}\right\|$$

scalar

$$= \left|\frac{\boldsymbol{u}\cdot\boldsymbol{a}}{\|\boldsymbol{a}\|^2}\right| \|\boldsymbol{a}\|$$ ← Theorem 3.2.1

$$= \frac{|\boldsymbol{u}\cdot\boldsymbol{a}|}{\|\boldsymbol{a}\|^2}\|\boldsymbol{a}\|$$ ← Since $\|\boldsymbol{a}\|^2 > 0$

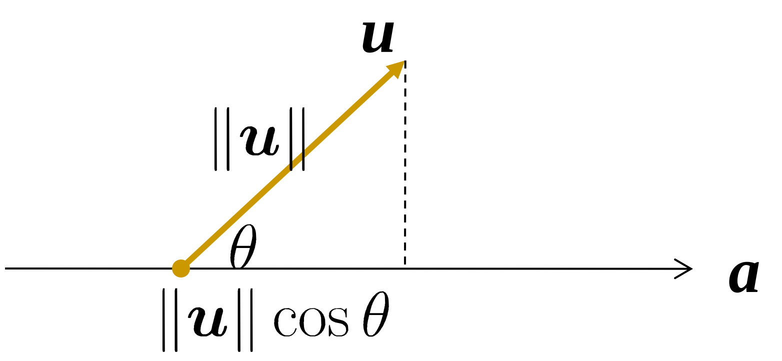
$$= \frac{|\boldsymbol{u}\cdot\boldsymbol{a}|}{\|\boldsymbol{a}\|}$$

If θ denotes the angle between ***u*** and ***a***, then $\boldsymbol{u}\cdot\boldsymbol{a} = \|\boldsymbol{u}\|\|\boldsymbol{a}\|\cos\theta$

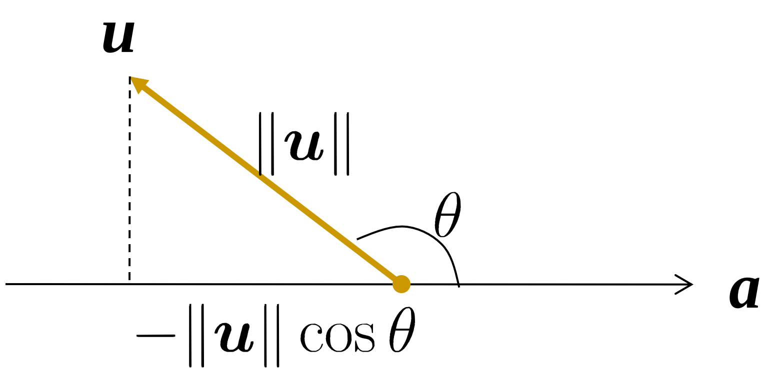
$$\|proj_{\boldsymbol{a}}\boldsymbol{u}\| = \|\boldsymbol{u}\||\cos\theta|$$

Length of Orthogonal Projection

$0 \leq \theta < \frac{\pi}{2}$



$\frac{\pi}{2} < \theta \leq \pi$



Theorem 3.3.3 Theorem of Pythagoras

- If **u** and **v** are orthogonal vectors in R^n with the Euclidean inner product, then

$$||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Proof:

Since **u** and **v** are orthogonal, $\mathbf{u} \cdot \mathbf{v}=0$, then

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 = (\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}+\boldsymbol{v}) = \|\boldsymbol{u}\|^2 + 2(\boldsymbol{u}\cdot\boldsymbol{v}) + \|\boldsymbol{v}\|^2$$
$$= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$$

Theorem 3.3.4

- (a) In R^2 the distance D between the point $P_0(x_0,y_0)$ and the line $ax+by+c=0$ is

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

- (b) In R^3 the distance D between the point $P_0(x_0,y_0,z_0)$ and the plane $ax+by+cz+d = 0$ is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Proof of Theorem 3.3.4(b)

- Let $Q(x_1,y_1,z_1)$ be any point in the plane. Position the normal **n**$=(a,b,c)$ so that its initial point is at Q.
- D is the length of the orthogonal projection of $\overrightarrow{QP_0}$ on **n**.

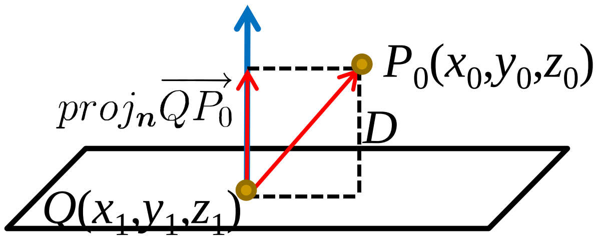
$$D = \|proj_{\boldsymbol{n}}\overrightarrow{QP_0}\| = \frac{|\overrightarrow{QP_0}\cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|}$$

$$\overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$$

$$\overrightarrow{QP_0}\cdot \boldsymbol{n} = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$$

$$\|\boldsymbol{n}\| = \sqrt{a^2 + b^2 + c^2}$$

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$



Proof of Theorem 3.3.4(b)

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

- Since the point $Q(x_1,y_1,z_1)$ lies in the given plane, $ax_1+by_1+cz_1+d = 0$, or $d=-ax_1-by_1-cz_1$
- Thus

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

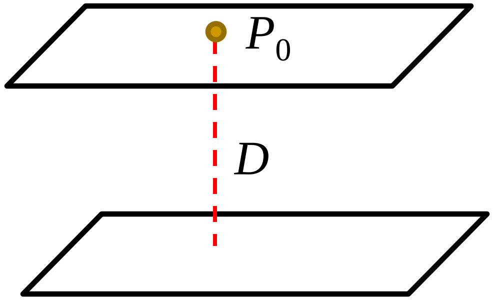
- Find the distance *D* from the point (1,-2) to the line 3*x*+4*y*-6 = 0 is

$$D = \frac{|3(1) + 4(-2) - 6|}{\sqrt{3^2 + 4^2}} = \frac{11}{5}$$

Distance Between Parallel Plane

- Two planes $x+2y-2z=3$ and $2x+4y-4z=7$
- To find the distance D between the planes, we can select an arbitrary point in one of the planes and compute its distance to the other plane.
- By setting $y=z=0$ in the equation $x+2y-2z=3$, we obtain the point $P_0(3,0,0)$ in this plane.
- The distance between P_0 and the plane $2x+4y-4z=7$ is

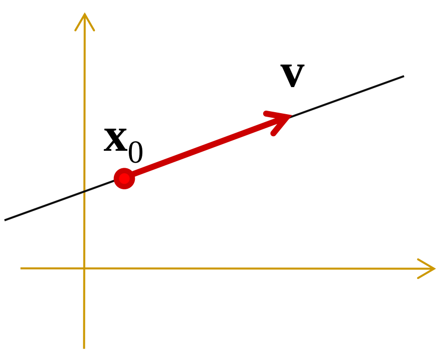
$$D = \frac{|2(3)+4(0)+(-4)(0)-7|}{\sqrt{2^2+4^2+(-4)^2}} = \frac{1}{6}$$

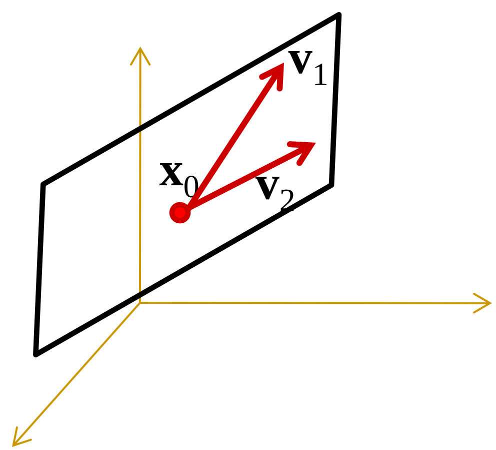


3.4
The Geometry of Linear Systems

Vector and Parametric Equations

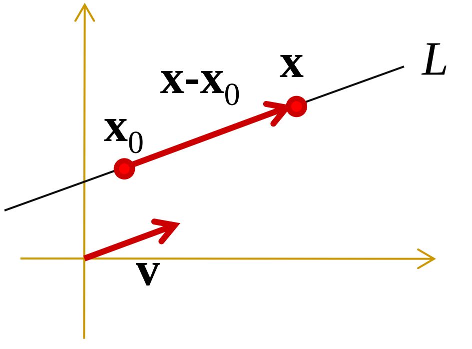
- A **unique line** in R^2 or R^3 is determined by a point $\mathbf{x}_0$ on the line and a nonzero vector $\mathbf{v}$ parallel to the line
- A **unique plane** in R^3 is determined by a point $\mathbf{x}_0$ in the plane and two ***noncollinear*** vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ parallel to the plane





Vector and Parametric Equations

- If **x** is a general point on such a line, the vector $\mathbf{x}-\mathbf{x}_0$ will be some scalar multiple of **v**
- $\mathbf{x}-\mathbf{x}_0 = t\mathbf{v}$ or equivalently $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$
- As the variable t (called ***parameter***) varies from $-\infty$ to ∞, the point **x** traces out the line L.



Theorem 3.4.1

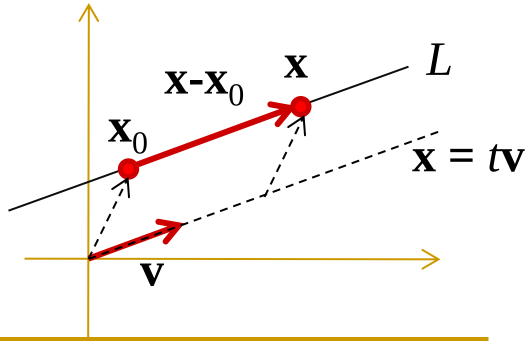
- Let L be the line in R^2 or R^3 that contains the point $\mathbf{x}_0$ and is parallel to the nonzero vector $\mathbf{v}$. Then the equation of the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$ is

$$\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$$

- If $\mathbf{x}_0=\mathbf{0}$, then the line passes through the origin and the equation has the form

$$\mathbf{x} = t\mathbf{v}$$

- The translation by $\mathbf{x}_0$ of the line through the origin

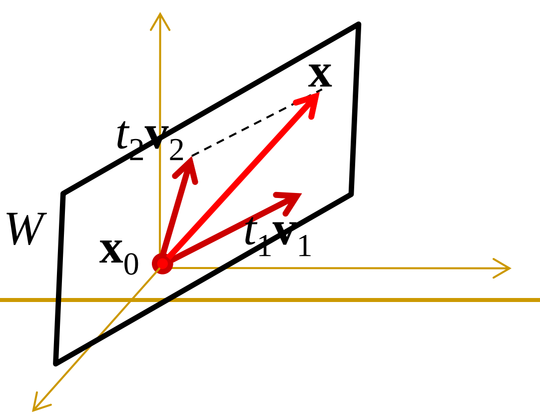


Vector and Parametric Equations

- If $\mathbf{x}$ is any point in the plane, then by forming suitable scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$, we can create a parallelogram with diagonal $\mathbf{x}-\mathbf{x}_0$ and adjacent sides $t_1\mathbf{v}_1$ and $t_2\mathbf{v}_2$. Thus we have

  $\mathbf{x} - \mathbf{x}_0 = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ or equivalently $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$

- As the variables t_1 and t_2 (*parameters*) vary independently from $-\infty$ to ∞, the point $\mathbf{x}$ varies over the entire plane W.



Theorem 3.4.2

- Let W be the plane in R^3 that contains the point $\mathbf{x}_0$ and is parallel to the noncollinear vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then an equation of the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$ is given by

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

- If $\mathbf{x}_0=\mathbf{0}$, then the plane passes through the origin and the equation has the form

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$$

Definition

- If $\mathbf{x}_0$ and $\mathbf{v}$ are vectors in R^n, and if $\mathbf{v}$ is nonzero, then the equation $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ defines **the line through $\mathbf{x}_0$ that is parallel to $\mathbf{v}$**. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to **pass through the origin**.
- If $\mathbf{x}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in R^n, and if $\mathbf{v}_1$ and $\mathbf{v}_2$ are not collinear, then the equation $\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ defines **the plane through $\mathbf{x}_0$ that is parallel to $\mathbf{v}_1$ and $\mathbf{v}_2$**. In the special case where $\mathbf{x}_0 = \mathbf{0}$, the line is said to **pass through the origin**.

Vector Forms

- The previous equations are called **vector forms** of a line and plane in R^n.
- If the vectors in these equations are expressed in terms of their components and the corresponding components on each side are equated, then the resulting equations are called **parametric equations** of the line and plane.

Example

- Find a vector equation and parametric equations of the line in R^3 that passes through the point $P_0(1,2,-3)$ and is parallel to the vector $\mathbf{v}=(4,-5,1)$
- Solution:

The line is $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$

If we let $\mathbf{x}=(x,y,z)$, and if we take $\mathbf{x}_0=(1,2,-3)$ then corresponding **the vector equation** is $(x,y,z)= (1,2,-3) + t(4,-5,1)$

Equating corresponding components on the two sides of this equation yields **the parametric equations** $x = 1+4t,\ y = 2-5t,\ z = -3+t$

Example

- Find vector and parametric equations of the plane $x - y + 2z = 5$
- Solution: solving for x in terms of y and z yields $x = 5 + y - 2z$
- Then using y and z as parameters t_1 and t_2, respectively, yields the parametric equations:

$$x = 5 + t_1 - 2t_2,\ y = t_1,\ z = t_2$$

- To obtain **a vector equation** of the plane we rewrite these parametric equations as $(x,y,z) = (5 + t_1 - 2t_2,\ t_1,\ t_2)$, or equivalently as $(x,y,z) = (5,0,0) + t_1(1,1,0) + t_2(-2,0,1)$

Example

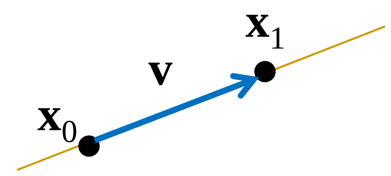
- Find vector and parametric equations of the plane in R^4 that passes through the point $\mathbf{x}_0=(2,-1,0,3)$ and is parallel to both $\mathbf{v}_1=(1,5,2,-4)$ and $\mathbf{v}_2=(0,7,-8,6)$
- Solution: the **vector equation** $\mathbf{x}=\mathbf{x}_0+t_1\mathbf{v}_1+t_2\mathbf{v}_2$ can be expressed as

$$(x_1,x_2,x_3,x_4) = (2,-1,0,3) + t_1(1,5,2,-4) + t_2(0,7,-8,6)$$

- Which yields the **parametric equations**

$$x_1 = 2+t_1,\ x_2 = -1+5t_1+7t_2,\ x_3 = 2t_1-8t_2,\ x_4=3-4t_1+6t_2$$

Lines Through Two points



- If $\mathbf{x}_0$ and $\mathbf{x}_1$ are distinct points in R^n, then the line determined by these points is parallel to the vector
  $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$
- The line can be expressed as $\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0)$
- Or equivalently as $\mathbf{x} = (1-t)\mathbf{x}_0 + t\mathbf{x}_1$
- These are called the ***two-point vector equations*** of a line in R^n

Example

- Find **vector** and **parametric** equations for the line in R^2 that passes through the points $P(0,7)$ and $Q(5,0)$
- Solution: Let's choose $\mathbf{x}_0=(0,7)$ and $\mathbf{x}_1=(5,0)$.
  $\mathbf{x}_1-\mathbf{x}_0 = (5,-7)$ and hence $(x,y) = (0,7) + t(5,-7)$
- We can rewrite in **parametric form** as $x = 5t$, $y = 7-7t$

Definition

- If $\mathbf{x}_0$ and $\mathbf{x}_1$ are vectors in R^n, then the equation $\mathbf{x} = \mathbf{x}_0 + t(\mathbf{x}_1 - \mathbf{x}_0)$ $(0 \leqq t \leqq 1)$ defines **the line segment from $\mathbf{x}_0$ to $\mathbf{x}_1$.**
- When convenient, it can be written as $\mathbf{x}=(1-t)\mathbf{x}_0 + t\mathbf{x}_1$ $(0 \leqq t \leqq 1)$
- Example: the line segment from $\mathbf{x}_0=(1,-3)$ to $\mathbf{x}_1=(5,6)$ can be represented by $\mathbf{x} = (1,-3) + t(4,9)$ $(0 \leqq t \leqq 1)$ or $\mathbf{x} = (1-t)(1,-3) + t(5,6)$ $(0 \leqq t \leqq 1)$

Dot Product Form of a Linear System

- Recall that a linear equation has the form

  $a_1x_1+a_2x_2+\ldots+a_nx_n = b$ $(a_1,a_2, \ldots,$ an not all zero)

- The corresponding homogeneous equation is

  $a_1x_1+a_2x_2+\ldots+a_nx_n = 0$ $(a_1,a_2, \ldots,$ an not all zero)

- These equations can be rewritten in vector form by letting

  $\mathbf{a} = (a_1,a_2,\ldots,a_n)$ and $\mathbf{x}=(x_1,x_2,\ldots,x_n)$

- Two equations can be written as

  $\boldsymbol{a} \cdot \boldsymbol{x} = b$ $\boldsymbol{a} \cdot \boldsymbol{x} = 0$

Dot Product Form of a Linear System

$$\boldsymbol{a} \cdot \boldsymbol{x} = 0$$

- It reveals that each solution vector **x** of a homogeneous equation is orthogonal to the coefficient vector **a**.
- Consider the homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

- If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1$, $\mathbf{r}_2$, …, $\mathbf{r}_m$, then we can write this system as

$$\begin{aligned} \boldsymbol{r}_1 \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{r}_m \cdot \boldsymbol{x} &= 0 \end{aligned}$$

Theorem 3.4.3

$$\begin{aligned} \boldsymbol{r}_1 \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{r}_2 \cdot \boldsymbol{x} &= 0 \\ &\vdots \\ \boldsymbol{r}_m \cdot \boldsymbol{x} &= 0 \end{aligned}$$

- If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x}=\mathbf{0}$ **consists of all vectors in R^n that are orthogonal to every row vector of A**.
- Example: the general solution of

Example 6 of Section 1.2

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is $x_1=-3r-4s-2t$, $x_2=r$, $x_3=-2s$, $x_4=s$, $x_5=t$, $x_6=0$

Vector form: $\mathbf{x} = (-3r-4s-2t, r, -2s, s, t, 0)$

Theorem 3.4.3

- According to Theorem 3.4.3, the vector **x** must be orthogonal to each of the row vectors

$\mathbf{r}_1 = (1,3,-2,0,2,0)$

$\mathbf{r}_2 = (2, 6, -5, -2, 4, -3)$

$\mathbf{r}_3 = (0,0,5,10,0,15)$

$\mathbf{r}_4 = (2,6,0,8,4,18)$

- Verify that $\mathbf{r}_1 \cdot \mathbf{x} =$ $1(-3r-4s-2t)+3(r)+(-2)(-2s)+0(s)+2(t)+0(0) = 0$

The Relationship Between $A\mathbf{x}=\mathbf{0}$ and $A\mathbf{x}=\mathbf{b}$

- Compare the solutions of the corresponding linear systems

Example 5 of Section 1.2

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 6 of Section 1.2

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

- Homogeneous system:
  x_1=-3r-4s-2t, x_2=r, x_3=-2s, x_4=s, x_5=t, x_6=0
- Nonhomogeneous system:
  x_1=-3r-4s-2t, x_2=r, x_3=-2s, x_4=s, x_5=t, x_6=1/3

The Relationship Between $A\mathbf{x}=\mathbf{0}$ and $A\mathbf{x}=\mathbf{b}$

- We can rewrite them in vector form:
  - Homogeneous system: $\mathbf{x} = (-3r-4s-2t, r, -2s, s, t, 0)$
  - Nonhomogeneous system: $\mathbf{x} = (-3r-4s-2t, r, -2s, s, t, 1/3)$
- By splitting the vectors on the right apart and collecting terms with like parameters,
  - Homogeneous system: $(x_1,x_2,x_3,x_4,x_5) = r(-3,1,0,0,0) + s(-4,0,-2,1,0,0) + t(-2,0,0,0,1,0)$
  - Nonhomogeneous system: $(x_1,x_2,x_3,x_4,x_5) = r(-3,1,0,0,0) + s(-4,0,-2,1,0,0) + t(-2,0,0,0,1,0) + (0,0,0,0,0,1/3)$
- **Each solution of the nonhomogeneous system can be obtained by adding (0,0,0,0,0,1/3) to the corresponding solution of the homogeneous system.**

Theorem 3.4.4

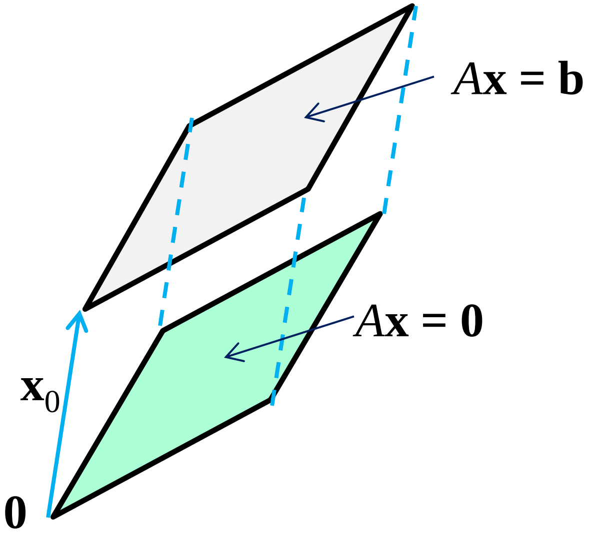
- The general solution of a consistent linear system $A\mathbf{x}=\mathbf{b}$ can be obtained by adding any specific solution of $A\mathbf{x}=\mathbf{b}$ to the general solution of $A\mathbf{x}=\mathbf{0}$.
- Proof:
- Let $\mathbf{x}_0$ be any specific solution of $A\mathbf{x}=\mathbf{b}$, Let ***W* denote the solution set** of $A\mathbf{x}=\mathbf{0}$, and let $\mathbf{x}_0+W$ denote the set of all vectors that result by adding $\mathbf{x}_0$ to each vector in W.
- Show that if $\mathbf{x}$ is a vector in $\mathbf{x}_0+W$, then $\mathbf{x}$ is a solution of $A\mathbf{x}=\mathbf{b}$, and conversely, that every solution of $A\mathbf{x}=\mathbf{b}$ is in the set $\mathbf{x}_0+W$.

Theorem 3.4.4

- Assume that $\mathbf{x}$ is a vector in $\mathbf{x}_0+W$. This implies that $\mathbf{x}$ is expressible in the form $\mathbf{x}=\mathbf{x}_0+\mathbf{w}$, where $A\mathbf{x}_0=\mathbf{b}$ and $A\mathbf{w}=\mathbf{0}$. Thus,
  $A\mathbf{x} = A(\mathbf{x}_0+\mathbf{w}) = A\mathbf{x}_0 + A\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}$
  which shows that $\mathbf{x}$ is a solution of $A\mathbf{x}=\mathbf{b}$.
- Conversely, let $\mathbf{x}$ be any solution of $A\mathbf{x}=\mathbf{b}$. To show that $\mathbf{x}$ is in the set $\mathbf{x}_0+W$ we must show that $\mathbf{x}$ is expressible in the form: $\mathbf{x} = \mathbf{x}_0+\mathbf{w}$, where $\mathbf{w}$ is in W ($A\mathbf{w} = \mathbf{0}$). We can do this by taking $\mathbf{w} = \mathbf{x}-\mathbf{x}_0$. It is in W since
  $A\mathbf{w} = A(\mathbf{x}-\mathbf{x}_0) = A\mathbf{x} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

Geometric Interpretation of Theorem 3.4.4

- We interpret vector addition as translation, then the theorem states that if $\mathbf{x}_0$ is any specific solution of $A\mathbf{x}=\mathbf{b}$, then the entire solution set of $A\mathbf{x}=\mathbf{b}$ can be obtained by translating the solution set of $A\mathbf{x}=\mathbf{0}$ by the vector $\mathbf{x}_0$.



3.5
Cross Product

Definition

- If **u** = (u_1,u_2,u_3) and **v**=(v_1,v_2,v_3) are vectors in 3-space, then the cross product (外積) **u**×**v** is the vector **defined by**

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$$

- Or, in determinant notation

$$\boldsymbol{u} \times \boldsymbol{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

- Remark: For the matrix $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$

  to find the first component of **u**×**v**, delete the first column and take the determinant, …

Example

- Find **u**×**v**, where **u**=(1,2,-2) and **v**=(3,0,1)
- Solution

$$\boldsymbol{u} \times \boldsymbol{v} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$
$$= (2, -7, -6)$$

Theorems

- Theorem 3.5.1 (Relationships Involving Cross Product and Dot Product)
  - If **u**, **v** and **w** are vectors in 3-space, then
    - $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$)
    - $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ ($\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$)
    - $\| \mathbf{u} \times \mathbf{v} \|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (**Lagrange's identity**)
    - $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\,\mathbf{w}$ (relationship between cross & dot product)
    - $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\,\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u}$ (relationship between cross & dot product)

Proof of Theorem 3.5.1(a)

Let $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{v}=(v_1,v_2,v_3)$

$$\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v})$$
$$=(u_1,u_2,u_3)\cdot(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$$
$$=u_1(u_2v_3-u_3v_2)+u_2(u_3v_1-u_1v_3)+u_3(u_1v_2-u_2v_1)=0$$

- Example: $\boldsymbol{u}=(1, 2, -2)$ and $\boldsymbol{v}=(3, 0, 1)$

$$\boldsymbol{u}\times\boldsymbol{v}=(2,-7,-6)$$
$$\boldsymbol{u}\cdot(\boldsymbol{u}\times\boldsymbol{v})=(1)(2)+(2)(-7)+(-2)(-6)=0$$
$$\boldsymbol{v}\cdot(\boldsymbol{u}\times\boldsymbol{v})=(3)(2)+(0)(-7)+(1)(-6)=0$$

Proof of Theorem 3.5.1(c)

$$\mathbf{u}\times\mathbf{v}=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$$

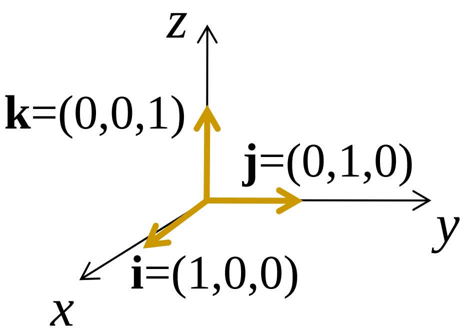
$$\|\boldsymbol{u}\times\boldsymbol{v}\|^2=(u_2v_3-u_3v_2)^2+(u_3v_1-u_1v_3)^2+(u_1v_2-u_2v_1)^2$$

$$\|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2-(\boldsymbol{u}\cdot\boldsymbol{v})^2$$
$$=(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2)-(u_1v_1+u_2v_2+u_3v_3)^2$$

Theorems

- Theorem 3.5.2 (Properties of Cross Product)
  - If **u**, **v** and **w** are any vectors in 3-space and k is any scalar, then
    - $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
    - $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
    - $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$
    - $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$
    - $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
    - $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
- Proof of (a)
  - Interchanging **u** and **v** interchanges the rows of the three determinants and hence changes the sign of each component in the cross product. Thus $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

Standard Unit Vectors



- The vectors

$$\mathbf{i} = (1,0,0),\ \mathbf{j} = (0,1,0),\ \mathbf{k} = (0,0,1)$$

  have length 1 and lie along the coordinate axes. They are called the standard unit vectors in 3-space.
- Every vector $\mathbf{v} = (v_1, v_2, v_3)$ in 3-space is expressible in terms of **i**, **j**, **k** since we can write

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1,0,0) + v_2\,(0,1,0) + v_3\,(0,0,1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

- For example, $(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$
- Note that

$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times i = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

Cross Product

- A cross product can be represented symbolically in the form of 3×3 determinant:

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & u_2 & u_3\\ v_1 & v_2 & v_3\end{vmatrix}=\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1 & u_3\\ v_1 & v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\mathbf{k}$$

- Example: if **u**=(1,2,-2) and **v**=(3,0,1)

$$\boldsymbol{u}\times\boldsymbol{v}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 1 & 2 & -2\\ 3 & 0 & 1\end{vmatrix}=2\boldsymbol{i}-7\boldsymbol{j}-6\boldsymbol{k}$$

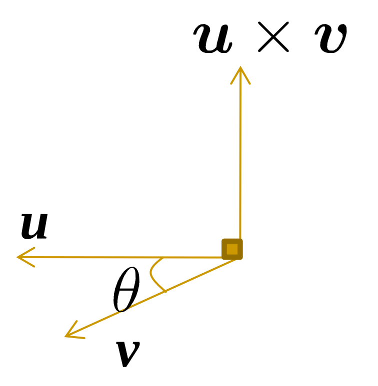
Cross Product

- It's not true in general that $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w}$
- For example:

$$\boldsymbol{i} \times (\boldsymbol{j} \times \boldsymbol{j}) = \boldsymbol{i} \times \boldsymbol{0} = \boldsymbol{0}$$

$$(\boldsymbol{i} \times \boldsymbol{j}) \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$$

- Right-hand rule
  - If the fingers of the right hand are cupped so they point in the direction of rotation, then the thumb indicates the direction of $\boldsymbol{u} \times \boldsymbol{v}$



Geometric Interpretation of Cross Product

- From Lagrange's identity, we have

$$\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$$

$$\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \cos^2\theta$$

$$= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2(1 - \cos^2\theta)$$

$$= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \sin^2\theta$$

- Since $0 \leq \theta \leq \pi$, it follows that $\sin\theta \geq 0$

so $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$

Geometric Interpretation of Cross Product

- From Lagrange's identity in Theorem 3.5.1

$$\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2$$

- If θ denotes the angle between **u** and **v**, then $\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \cos\theta$

$$\begin{aligned}&\|\boldsymbol{u} \times \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 - \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \cos^2\theta \\ &= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2(1 - \cos^2\theta) \\ &= \|\boldsymbol{u}\|^2\|\boldsymbol{v}\|^2 \sin^2\theta\end{aligned}$$

- Since $0 \le \theta \le \pi$, it follows that $\sin\theta \ge 0$, thus

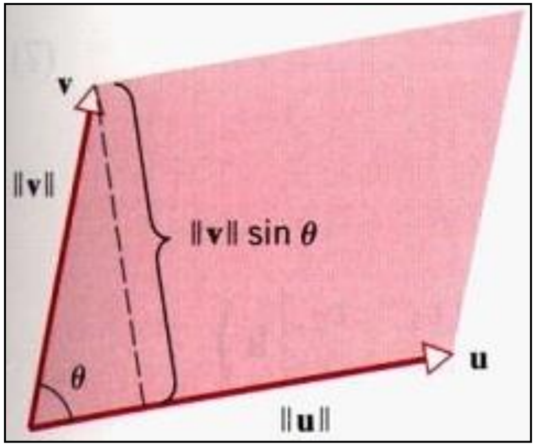
$$\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$$

Geometric Interpretation of Cross Product

$$\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta$$

- $\|\boldsymbol{v}\|\sin\theta$ is the altitude (頂垂線) of the parallelogram determined by **u** and **v**. Thus, the area *A* of this parallelogram is given by

$$A = \|\boldsymbol{u}\|\|\boldsymbol{v}\|\sin\theta = \|\boldsymbol{u}\times\boldsymbol{v}\|$$

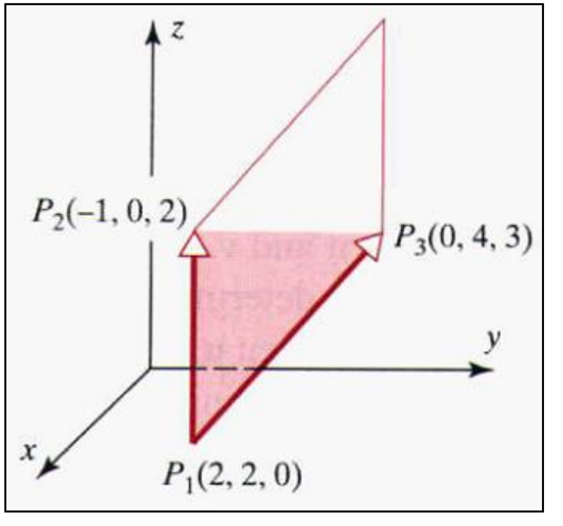


- This result is even correct if **u** and **v** are collinear, since we have $\|\boldsymbol{u}\times\boldsymbol{v}\| = \mathbf{0}$ when $\theta = 0$

Area of a Parallelogram

- Theorem 3.5.3 (Area of a Parallelogram)
  - If **u** and **v** are vectors in 3-space, then $\|\mathbf{u} \times \mathbf{v}\|$ is equal to the area of the parallelogram determined by **u** and **v**.

- Example
  - Find the area of the triangle determined by the point (2,2,0), (-1,0,2), and (0,4,3).



$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-3, -2, 2) \times (-2, 2, 3)$$
$$= (-10, 5, -10)$$
$$A = \tfrac{1}{2}\|\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}\| = \tfrac{1}{2}(15) = \tfrac{15}{2}$$

Triple Product

- Definition
  - If **u**, **v** and **w** are vectors in 3-space, then **u · (v × w)** is called the scalar triple product (純量三乘積) of **u**, **v** and **w**.

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$\boldsymbol{u}\cdot(\boldsymbol{v}\times\boldsymbol{w}) = \boldsymbol{u}\cdot\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\boldsymbol{k}\right)$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Example

- **u** = 3**i** – 2**j** – 5**k**, **v** = **i** + 4**j** – 4**k**, **w** = 3**j** + 2**k**

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 4 & -4 \\ 3 & 3 \end{vmatrix} - (-2) \begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5) \begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 60 + 4 - 15 = 49$$

Triple Product

- Remarks:
  - The symbol $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$ make no sense because we cannot form the cross product of a scalar and a vector.
  - $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$, since the determinants that represent these products can be obtained from one another by *two* row interchanges.

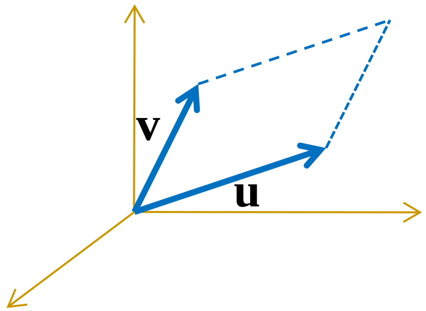
$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \qquad \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

Theorem 3.5.4

- The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}$ is equal to the area of the parallelogram in 2-space determined by the vectors $\mathbf{u} = (u_1, u_2)$, and $\mathbf{v} = (v_1, v_2)$,
- The absolute value of the determinant $\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$ is equal to the volume of the parallelepiped in 3-space determined by the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$,

Proof of Theorem 3.5.4(a)



- View **u** and **v** as vectors in the *xy*-plane of an *xyz*-coordinate system. Express **u**=(u_1,u_2,0) and **v**=(v_1,v_2,0)

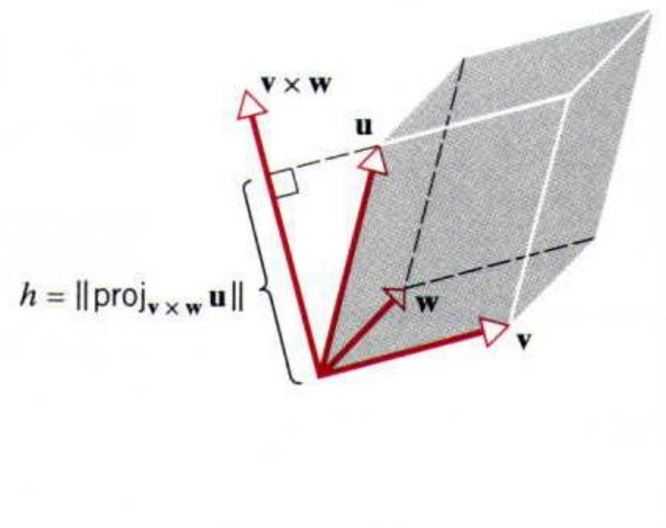
$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{k} = \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \boldsymbol{k}$$

- It follows from Theorem 3.5.3 and the fact that $\|\boldsymbol{k}\| = 1$ that the area *A* of the parallelogram determined by **u** and **v** is

$$A = \|\boldsymbol{u} \times \boldsymbol{v}\| = \left\| \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \boldsymbol{k} \right\| = \left| \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right| \|\boldsymbol{k}\|$$

$$= \left| \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \right|$$

Proof of Theorem 3.5.4(b)



- The area of the base is $\|\boldsymbol{v} \times \boldsymbol{w}\|$
- The height *h* of the parallelepiped is the length of the orthogonal projection of **u** on $\boldsymbol{v} \times \boldsymbol{w}$

$$h = \|proj_{\boldsymbol{v} \times \boldsymbol{w}} \boldsymbol{u}\| = \frac{|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|}{\|\boldsymbol{v} \times \boldsymbol{w}\|}$$

- The volume *V* of the parallelepiped is

$$V = \|\boldsymbol{v} \times \boldsymbol{w}\| \frac{|\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|}{\|\boldsymbol{v} \times \boldsymbol{w}\|} = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$$

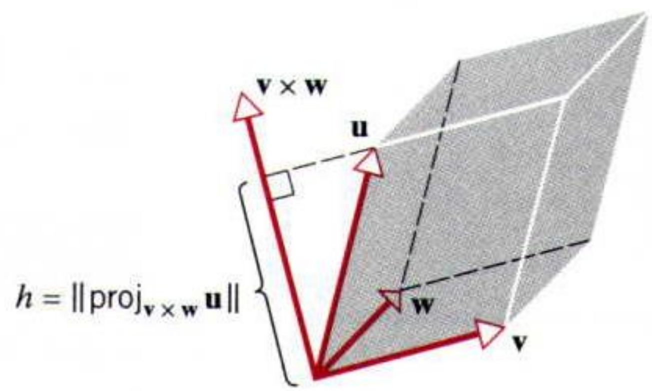
$$\|proj_{\boldsymbol{a}} \boldsymbol{u}\| = \frac{|\boldsymbol{u} \cdot \boldsymbol{a}|}{\|\boldsymbol{a}\|}$$

Remark

$$V = \left|\det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}\right|$$

$$V = \begin{bmatrix} \text{volume of parallelepiped} \\ \text{determined by } \mathbf{u}, \mathbf{v}, \text{ and } \mathbf{w} \end{bmatrix} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



(b)

Remark

$$V = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$$

- We can conclude that

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \pm V$$

where + or – results depending on whether **u** makes an acute or an obtuse angle with $\boldsymbol{v} \times \boldsymbol{w}$

Theorem 3.5.5

- If the vectors **u** = (u_1, u_2, u_3), **v** = (v_1, v_2, v_3), and **w** = (w_1, w_2, w_3) have the same initial point, then they lie in the same plane if and only if

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$