Chapter 2
Determinants

Outline

- 2.1 Determinants by Cofactor Expansion
- 2.2 Evaluating Determinants by Row Reduction
- 2.3 Properties of Determinants; Cramer's Rule

2.1

Determinants by Cofactor Expansion

Determinant

- Recall from Theorem 1.4.5 that the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad ? \Rightarrow \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  is invertible if $ad - bc \neq 0$. It is called the ***determinant*** (行列式) of the matrix *A* and is denoted by the symbol det(*A*) or |*A*|

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Minor and Cofactor

- Definition
  - Let A be $n \times n$
    - The (i,j)-minor (子行列式) of A, denoted M_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jth column from A
    - The (i,j)-cofactor (餘因子) of A, denoted C_{ij}, is $(-1)^{i+j} M_{ij}$
- Remark
  - Note that $C_{ij} = \pm M_{ij}$ and the signs $(-1)^{i+j}$ in the definition of cofactor form a checkerboard pattern:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Example

- Let
$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$
- The minor of entry a_{11} is $M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$
- The cofactor of a_{11} is $C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$
- Similarly, the minor of entry a_{32} is $M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$
- The cofactor of a_{32} is $C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$

Cofactor Expansion of a 2 x 2 Matrix

- For the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$C_{11} = M_{11} = a_{22} \qquad C_{12} = -M_{12} = -a_{21}$$

$$C_{21} = -M_{21} = -a_{12} \qquad C_{22} = M_{22} = a_{11}$$

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}C_{11} + a_{12}C_{12} \\ &= a_{21}C_{21} + a_{22}C_{22} \\ &= a_{11}C_{11} + a_{21}C_{21} \\ &= a_{12}C_{12} + a_{22}C_{22} \end{aligned}$$

These are called cofactor expansions of *A*

Cofactor Expansion

- Theorem 2.1.1 (Expansions by Cofactors)
  - The **determinant** of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products; that is, for each $1 \leq i, j \leq n$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}$$

(cofactor expansion along the jth column)

and

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$$

(cofactor expansion along the ith row)

- Example

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix} = 3(-4) - (-2)(-2) + 5(3) = -1$$

(cofactor expansion along the first column)

Example

- Cofactor expansion along the first row

$$\begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

Example

- Smart choice of row or column

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

- It's easiest to use **cofactor expansion** along the second column

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1 \cdot (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6$$

Determinant of an Lower Triangular Matrix

- For simplicity of notation, we prove the result for a 4×4 lower triangular matrix

$$A = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

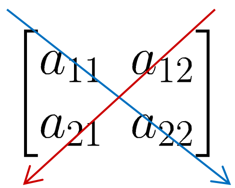
$$\det(A) = \begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \left| a_{44} \right| = a_{11}a_{22}a_{33}a_{44}$$

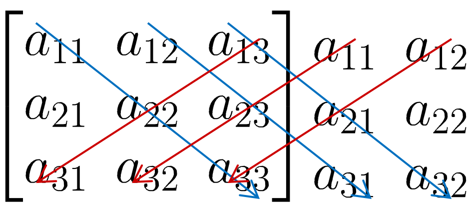
Theorem 2.1.2

- If A is an $n \times n$ triangular matrix, then det(A) is the product of the entries on the main diagonal of the matrix: $\det(A) = a_{11}a_{22}\cdots a_{nn}$

Useful Technique for 2x2 and 3x3 Matrices



$$\det = a_{11}a_{22} - a_{12}a_{21}$$



First, recopy the first and second columns as shown in the figure. After that, compute the determinant by summing the products of entries on the rightward arrows and subtracting the products on the leftward arrows.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

2.2

Evaluating Determinants by Row Reduction

Theorem 2.2.1

- Let *A* be a square matrix. If *A* has a row of zeros or a column of zeros, then det(*A*) = 0.
- Proof:
  - Since the determinant of *A* can be found by a cofactor expansion along any row or column, we can use the row or column of zeros.

$$\det(A) = 0C_1 + 0C_2 + \cdots + 0C_n = 0$$

Theorem 2.2.2

- Let A be a square matrix. Then $\det(A) = \det(A^T)$
- Proof:
  - Since transposing a matrix changes it columns to rows and its rows to columns, the cofactor expansion of A along any row is the same as the cofactor expansion of A^T along the corresponding column. Thus, both have the same determinant.

Theorem 2.2.3 (Elementary Row Operations)

- Let A be an $n \times n$ matrix
  - If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, than $\det(B) = k \det(A)$
  - If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = - \det(A)$
  - If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple column is added to another column, then $\det(B) = \det(A)$

Example

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = ka_{11}C_{11} + ka_{12}C_{12} + ka_{13}C_{13}$$

$$= k(a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}) = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \stackrel{?}{=} - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \stackrel{?}{=} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Theorems

- Theorem 2.2.4 (Elementary Matrices)
  - Let E be an $n \times n$ elementary matrix (基本矩陣)
    - If E results from multiplying a row of I_n by k, then $\det(E) = k$
    - If E results from interchanging two rows of I_n, then $\det(E) = -1$
    - If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3 \qquad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1 \qquad \begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

Theorems

- Theorem 2.2.5 (Matrices with Proportional Rows or Columns)
  - If A is a square matrix with two proportional rows or two proportional column, then det(A) = 0

-2 times Row 1 was added to Row 2

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

$$\begin{bmatrix} -1 & 4 \\ -2 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 7 \\ -4 & 8 & 5 \\ 2 & -4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 & 4 & -5 \\ 6 & -2 & 5 & 2 \\ 5 & 8 & 1 & 4 \\ -9 & 3 & -12 & 15 \end{bmatrix}$$

Example (Using Row Reduction to Evaluate a Determinant)

- Evaluate det(A) where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

- Solution:

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = -\begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← The first and second rows of A are interchanged.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign

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Example

$$\det(A) = -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

← -2 times the first row was added to the third row.

$$= -3\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

← -10 times the second row was added to the third row

$$= (-3)(-55)\begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

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Example

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

- Using column operations to evaluate a determinant
- Put *A* in **lower triangular form** by adding -3 times the first column to the fourth to obtain

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix} = (1)(7)(3)(-26) = -546$$

Example

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Using Row Operations & Cofactor Expansion

- By adding suitable multiples of the second row to the remaining rows, we obtain

Cofactor expansion along the first column

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix} = -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} = -18$$

Add the first row to the third row

Cofactor expansion along the first column

2.3

Properties of Determinants; Cramer's Rule

Basic Properties of Determinant

- Since a common factor of any row of a matrix can be moved through the det sign, and since each of the n row in kA has a common factor of k, we obtain

$$\det(kA) = k^n \det(A)$$

- There is no simple relationship exists between $\det(A)$, $\det(B)$, and $\det(A+B)$ in general.
- In particular, we emphasize that $\det(A+B)$ is usually *not* equal to $\det(A) + \det(B)$.

Example

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Consider

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad A + B = \begin{bmatrix} 4 & 3 \\ 3 & 8 \end{bmatrix}$$

- We have det(A) = 1, det(B) = 8, and det(A+B)=23; thus

$$\det(A + B) \neq \det(A) + \det(B)$$

Example

- Consider two matrices that differ only in the second row

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\det(A) + \det(B) = (a_{11}a_{22} - a_{12}a_{21}) + (a_{11}b_{22} - a_{12}b_{21})$$

$$= a_{11}(a_{22} + b_{22}) - a_{12}(a_{21} + b_{21})$$

$$= \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \det \begin{bmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{bmatrix} = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

Theorems 2.3.1

- Let A, B, and C be $n \times n$ matrices that **differ only in a single row**, say the r-th, and assume that the r-th row of C can be obtained by adding corresponding entries in the r-th rows of A and B. Then

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

- Example

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Theorems

- Lemma 2.3.2
  - If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

- Remark:
  - If B is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$, are $n \times n$ elementary matrices, then

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2) \cdots \det(E_r)\det(B)$$

Proof of Lemma 2.3.2

If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then
$$\det(EB) = \det(E)\det(B)$$

- We shall consider three cases, each depending on the row operation that produces matrix E.
- Case 1. If E results from multiplying a row of I_n by k, then by Theorem 1.5.1, EB results from B by multiplying a row by k; so from Theorem 2.2.3a we have
$$\det(EB) = k\det(B)$$
From Theorem 2.2.4a, we have $\det(E) = k$, so
$$\det(EB) = \det(E)\det(B)$$
- Cases 2 and 3. E results from interchanging two rows of I_n or from adding a multiple of one row to another.

Theorems

- **Theorem 2.3.3 (Determinant Test for Invertibility)**
  - A square matrix A is invertible if and only if $\det(A) \neq 0$
- Proof: Let R be the reduced row-echelon form of A.

$$R = E_r \cdots E_2 E_1 A$$

$$\det(R) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(A)$$

From Theorem 2.2.4, the determinants of the elementary matrices are all nonzero. Thus, $\det(A)$ and $\det(R)$ are both zero or both nonzero.

Proof of Theorem 2.3.3

- If A is invertible, then by Theorem 1.6.4, we have $R = I$, so $\det(R) = 1 \neq 0$ and consequently $\det(A) \neq 0$.
- Conversely, if $\det(A) \neq 0$, then $\det(R) \neq 0$, so R cannot have a row of zeros. It follows from Theorem 1.4.3 that $R=I$, so A is invertible by Theorem 1.6.4.

Example: Determinant Test for Invertibility

- Since the first and third rows are proportional, det(A) = 0

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

- A is not invertible.

Theorems

- **Theorem 2.3.4**
  - If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

- **Theorem 2.3.5**
  - If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof of Theorem 2.3.4

- **If the matrix *A* is not invertible**, then by Theorem 1.6.5 neither is the product *AB*.
- Thus, from Theorem 2.3.3, we have det(*AB*) = 0 and det(*A*) = 0, so it follows that det(*AB*) = det(*A*) det(*B*).
- **Now assume that *A* is invertible**. By Theorem 1.6.4, the matrix *A* is expressible as a product of elementary matrices, say

$$A = E_1 E_2 \cdots E_r$$

$$AB = E_1 E_2 \cdots E_r B$$

Proof of Theorem 2.3.4

$$AB = E_1 E_2 \cdots E_r B$$

⇓

$$\det(AB) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

⇓

$$\det(AB) = \det(E_1 E_2 \cdots E_r)\det(B)$$

⇓

$$\det(AB) = \det(A)\det(B)$$

Proof of Theorem 2.3.5

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

- Since $A^{-1}A = I$, it follows that $\det(A^{-1}A)=\det(I)$.
- Therefore, we must have $\det(A^{-1})\det(A) = 1$.
- Since $\det(A) \neq 0$, the proof can be completed by dividing through by $\det(A)$.

Example

- If one multiplies the entries in any row by the corresponding cofactors from a ***different*** row, the sum of these products is always zero.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Consider the quantity $a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = ?$
- Construct a new matrix *A*' by replacing the third row of *A* with another copy of the first row

$$A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A' = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

- Since the first two rows of A and A' are the same, and since the computations of C_{31}, C_{32}, C_{33}, C_{31}', C_{32}', and C_{33}' involve only entries from the first two rows of A and A', it follows that

$$C_{31} = C'_{31} \qquad C_{32} = C'_{32} \qquad C_{33} = C'_{33}$$

- Since A' has two identical rows, det(A') = 0
- By evaluating det(A') by cofactor expansion along the third row gives

$$\det(A') = a_{11}C'_{31} + a_{12}C'_{32} + a_{13}C'_{33} = a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0$$

Definition

- If A is any $n \times n$ matrix, and C_{ij} is the cofactor of a_{ij}, then the matrix is called the ***matrix of cofactors from A*** (餘因子矩陣).

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

- The transpose of this matrix is called the ***adjoint of A*** (伴隨矩陣) and is denoted by adj(A)

$$\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Adjoint of a 3x3 Matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Cofactors of *A* are

$$C_{11} = 12 \quad C_{12} = 6 \quad C_{13} = -16$$
$$C_{21} = 4 \quad C_{22} = 2 \quad C_{23} = 16$$
$$C_{31} = 12 \quad C_{32} = -10 \quad C_{33} = 16$$

The matrix of cofactors is

$$\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$$

The adjoint of *A*

$$\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorems

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$A\,\mathrm{adj}(A) = \det(A)\,I$$

- Theorem 2.3.6 (Inverse of a Matrix using its Adjoint)
  - If A is an invertible matrix, then $A^{-1} = \dfrac{1}{\det(A)}\mathrm{adj}(A)$

Proof of Theorem 2.3.6

If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$

- We show first that $A\text{adj}(A) = \det(A)I$

$$Aadj(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{j1} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{j2} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix}$$

- The entry in the ith row and jth column of $A\text{adj}(A)$ is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

Proof of Theorem 2.3.6

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

- If $i=j$, then it is the cofactor expansion of det(A) along the ith row of A.
- If $i \neq j$, then the a's and the cofactors come from different rows of A, so the value is zero. Therefore,

$$Aadj(A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

- Since A is invertible, det(A)$\neq$ 0. Therefore

$$\frac{1}{\det(A)}[Aadj(A)] = I \Rightarrow A\left[\frac{1}{\det(A)}adj(A)\right] = I$$

- Multiplying both sides on the left by A^{-1} yields $A^{-1} = \frac{1}{\det(A)}adj(A)$

Example

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \qquad \text{The adjoint of } A = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} adj(A) = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Theorem 2.3.7 (Cramer's Rule)

- If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \cdots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix $\mathbf{b} = [b_1 \; b_2 \; \cdots \; b_n]^T$

Proof of Theorem 2.3.7

- If $\det(A) \neq 0$, then *A* is invertible, and by Theorem 1.6.2, $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ is the unique solution of $A\boldsymbol{x} = \boldsymbol{b}$. Therefore, by Theorem 2.3.6, we have

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \frac{1}{\det(A)}\text{adj}(A)\boldsymbol{b}$$

$$= \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Proof of Theorem 2.3.7

$$\boldsymbol{x} = \frac{1}{\det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

- The entry in the *j*th row of ***x*** is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

- Now let

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

Proof of Theorem 2.3.7

- Since A_j differs form A only in the jth column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A.
- The cofactor expansion of $\det(A_j)$ along the jth column is therefore $\det(A_j) = b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}$
- Substituting this result gives

$$x_j = \frac{\det(A_j)}{\det(A)}$$

Example

- Use Cramer's rule to solve

$$\begin{aligned} x_1 + \qquad + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

- Since

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

- Thus,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11}, x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Theorem 2.3.8 (Equivalent Statements)

- If A is an $n \times n$ matrix, then the following are equivalent
  - A is invertible.
  - $A\mathbf{x} = \mathbf{0}$ has only the trivial solution
  - The reduced row-echelon form of A as I_n
  - A is expressible as a product of elementary matrices
  - $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$
  - $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$
  - $\det(A) \neq 0$