Propagation of Error

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Reference:
1. W. Navidi. *Statistics for Engineering and Scientists*. Chapter 3 & Teaching Material

Introduction

- Any measuring procedure contains error
  - This causes measured values to differ from the true values that are measured
- Errors in measurement produce error in calculated values (like the mean)
- Definition: When error in measurement produces error in calculated values, we say that error is propagated from the measurements to the calculated value
- Having knowledge concerning the sizes of the errors in measurement $\rightarrow$ Obtaining knowledge concerning the likely size of the error in a calculated quantity

Measurement Error

- A geologist weighs a rock on a scale and gets the following measurements:

  251.3 252.5 250.8 251.1 250.4

- None of the measurements are the same and none are probably the actual measurement

- The error in the measured value is the difference between a measured value and the true value

Parts of Error

- We think of the error of the measurement as being composed of two parts:
  - Systematic error (or bias)
  - Random error
- Bias is error that is the same for every measurement
  - E.g., a imperfectly calibrated scale always gives you a reading that is too low
- Random error is error that varies from measurement to measurement and averages out to zero in the long run
  - E.g., Parallax (視差) error

$$\mathbf{E}[\text{Random Error}] = 0$$

  - The difference in the position of dial indicator when observed from different angles

$$\underbrace{\text{Measured Value}}_{\text{random variable}} = \underbrace{\text{True Value}}_{\text{constant}} + \underbrace{\text{Bias}}_{\text{constant}} + \underbrace{\text{Random Error}}_{\text{random variable}}$$

Two Aspects of the Measuring Process (1/4)

- We are interested in accuracy
  - Accuracy is determined by bias
    - The bias in the measuring process is the difference between the mean measurement μ and the true value:

      bias = μ - true value
  - The smaller the bias, the more accurate the measuring process
    - Unbiased: the mean measurement is equal to the true value
- The other aspect is precision
  - Precision refers to the degree to which repeated measurements of the same quantity tend to agree with each other
  - If repeated measurements come out nearly the same every time, the precision is high
  - The uncertainty in the measuring process is the standard deviation σ
    - The smaller the uncertainty, the more precise the measuring process

Two Aspects of the Measuring Process (2/4)

- Example: Figures 3.1 and 3.2

both bias and uncertainty are small

bias is small; uncertainty is large

bias is large; uncertainty is small

both bias and uncertainty are large

How about bias? (True value is often unknown in real life)

uncertainty is small

uncertainty is large

uncertainty is small

uncertainty is large

Two Aspects of the Measuring Process (3/4)

- Let $X_1, \ldots, X_n$ be independent measurements, all made by the same process on the same quantity
  - The sample standard deviation *s* can be used to estimate the uncertainty
  - Estimates of uncertainty are often crude, especially when based on small samples
  - If the true value is know, the sample mean, $\overline{X}$, can be used to estimate the bias:

$$\text{bias} \approx \overline{X} - \text{true value}$$

  - If the true value is unknown, the bias cannot be estimated from repeated measurements

Two Aspects of the Measuring Process (4/4)

- From now on, we will describe measurements in the form

$$\text{Measured value} \pm \sigma$$

  - Where σ is the uncertainty in the process that produced the measured value
  - Assume that bias has reduced to a negligible level

Linear Combinations of Measurements

- Each measurement can be viewed as a random variable
  - Its standard deviation represents the uncertainty for it
- How to compute uncertainties in scaled measurements or combinations of independent measurements?
  - If X is a measurement and c is a constant, then

$$\sigma_{cX} = |c|\sigma_X$$

  - If $X_1, \ldots, X_n$ are independent measurements and $c_1, \ldots, c_n$ are constants, then

$$\sigma_{c_1X_1+\cdots+c_nX_n} = \sqrt{c_1^2\sigma_{X_1}^2 + \cdots + c_n^2\sigma_{X_n}^2}$$

Example 3.6

- **Question:** A surveyor is measuring the perimeter of a rectangular lot. He measures two adjacent sides to be 50.11 ± 0.05m and 75.12 ± 0.08m. These measurements are independent. Estimate the perimeter of the lot and find the uncertainty in the estimate.

- **Answer:** Let $X = 50.11$ and $Y = 75.21$ be the two measurements. The perimeter is estimated by $P = 2X + 2Y = 250.64$m, and the uncertainty in P is
  $\sigma_P = \sigma_{2X+2y} = \sqrt{4\sigma_X^2 + 4\sigma_Y^2} = \sqrt{4(0.05)^2 + 4(0.08)^2} = 0.19\text{m}$
  So the perimeter is 250.64 ± 0.19m.

Repeated Measurements

- One of the best ways to reduce uncertainty is to take independent measurements and average them (why?)
  - The measurements can be viewed as a simple random sample from a population
  - Their average is therefore the sample mean
- If $X_1, \ldots, X_n$ are n *independent* measurements, each with mean μ and standard deviation σ, then the sample mean, $\overline{X}$, is a measurement with mean (cf. Sections 2.5 & 2.6)

$$\mu_{\overline{X}} = \mu$$

and with uncertainty

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}}$$

The uncertainty is reduced by a factor equal to the square root of the sample size

Example 3.8

- **Question:** The length of a component is to be measured by a process whose uncertainty is 0.05 cm. If 25 independent measurements are made and the average of these is used to estimate the length, what will the uncertainty be? How much more precise is the average of 25 measurements than a single measurement?

Answer: The uncertainty is $0.05/\sqrt{25} = 0.01$ cm. The average of 25 independent measurements is five times more precise than a single measurement.

Repeated Measurements with Differing Uncertainties (1/2)

- It happens that the repeated measurements are made with different instruments independently
  - A more suitable way is to calculate a weighted average of the measurements instead of the sample mean of them

$$\sigma_{w-avg} = \sigma_{w_1 X_1 + \cdots + w_n X_n} = \sqrt{w_1^2 \sigma_{X_1}^2 + \cdots + w_i^2 \sigma_{X_i}^2 + \cdots + w_n^2 \sigma_{X_n}^2}$$

where $0 \leq w_i \leq 1$ and $\sum_i w_i = 1$

Repeated Measurements with Differing Uncertainties (2/2)

- **Special Case:** If X and Y are *independent* measurements of the same quantity, with uncertainties σ_X and σ_Y, respectively, then the weighted average of X and Y with the smallest uncertainty is given by ,
where

$$w_{best} = \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} \qquad 1 - w_{best} = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_Y^2}$$

$$\sigma_{w-avg} = \sqrt{w^2 \sigma_x^2 + (1-w)^2 \sigma_y^2}$$

$w^2 \sigma_x^2 + (1-w)^2 \sigma_y^2$ has the minimum value when

$$\frac{d\left(w^2 \sigma_x^2 + (1-w)^2 \sigma_y^2\right)}{dw} = 0$$

$$w \sigma_x^2 - (1-w) \sigma_y^2 = 0$$

$$\therefore\ w = \frac{\sigma_y^2}{\sigma_X^2 + \sigma_y^2}$$

Linear Combinations of Dependent Measurements

- If $X_1, \ldots, X_n$ are measurements and $c_1, \ldots, c_n$ are constants, then

$$\sigma_{c_1 X_1 + \cdots + c_n X_n} \le |c_1| \sigma_{X_1} + \cdots + |c_n| \sigma_{X_n}$$

  - $|c_1| \sigma_{X_1} + \cdots + |c_n| \sigma_{X_n}$ is a conservative estimate of the uncertainty in $c_1 X_1 + \cdots + c_n X_n$
  - For a detailed proof, refer to the textbook (p.176)

Example 3.13

- **Question:** A surveyor is measuring the perimeter of a rectanglar lot. Two adjacent sides are measured as 50.11 ± 0.05 m and 75.21 ± 0.08 m, respectively. These measurements are not necessarily independent. Find a conservative estimate of the uncertainty in the measured value of the perimeter.

- **Answer:** Two measurements are denoted by X_1 and X_2 with uncertainties $\sigma_{X_1} = 0.05$ and $\sigma_{X_2} = 0.08$, respectively. Let the perimeter be given by $P = 2X_1 + 2X_2$. The corresponding uncertainty of P is therefore constrained by

$$\sigma_P = \sigma_{2X_1+2X_2} \le 2\sigma_{X_1} + 2\sigma_{X_2} = 2(0.05) + 2(0.08) = 0.26$$

Uncertainties for (Nonlinear) Functions of One Measurement (1/4)

- If X is a measurement whose uncertainty σ_X is small, and if U is a (nonlinear) function of X, then

$$\sigma_U \approx \left|\frac{dU}{dX}\right| \sigma_X$$ Equation (3.10)

  - This is the propagation of error formula
  - The derivative $\left|\frac{dU}{dX}\right|$ is evaluated at the observed measurement X
    - A constant value!

Uncertainties for (Nonlinear) Functions of One Measurement (2/4)

- Taylor series approximation (linearizing) of U

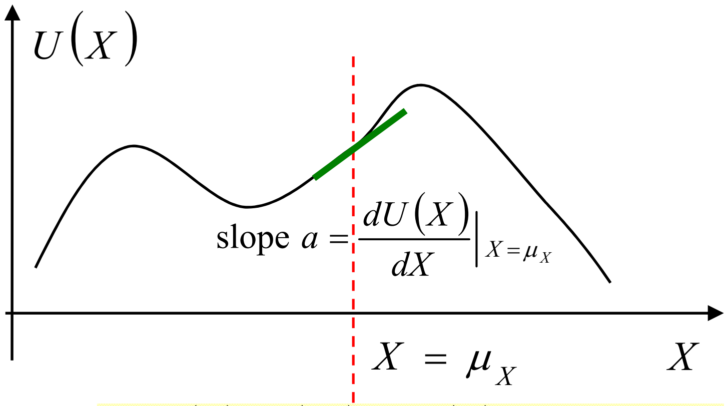
$$U = U(X) \approx U(\mu_X) + \left(\frac{dU(X)}{dX} \bigg|_{X=\mu_X} \right)(X - \mu_X)$$

$$= \underbrace{U(\mu_X) - \left(\frac{dU(X)}{dX} \bigg|_{X=\mu_X} \right)\mu_X}_{\text{constant}} + \left(\frac{dU(X)}{dX} \bigg|_{X=\mu_X} \right) X$$

$$\Rightarrow \quad \sigma_U = \left| \frac{dU(X)}{dX} \bigg|_{X=\mu_X} \right| \sigma_X$$

$$= \left| \frac{dU}{dX} \right| \sigma_X$$

simply denoted as



$$a = \frac{U(X) - U(\mu_X)}{X - \mu_X} = \frac{dU(X)}{dX}\bigg|_{X=\mu_X}$$

$$\Rightarrow U(X) = U(\mu_X) + \left(\frac{dU(X)}{dX}\bigg|_{X=\mu_X} \right)(X - \mu_X)$$

Uncertainties for (Nonlinear) Functions of One Measurement (3/4)

- **Definition:** If U is a measurement whose true value is μ_U, and whose uncertainty is σ_U, the relative uncertainty in U is the quantity

$$\frac{\sigma_U}{\mu_U}$$

- However, in practice, μ_U is unknown, so the relative uncertainty is estimated by

$$\frac{\sigma_U}{U}$$

- The relative uncertainty is also called the "coefficient of variation"

Uncertainties for (Nonlinear) Functions of One Measurement (4/4)

- There are two methods for approximating the relative uncertainty σ_U / U in a function $U = U(X)$
  1. Compute σ_U using Equation (3.10) and then divide by U
  2. Compute $\ln U$ and use equation Equation (3.10) to find $\sigma_{\ln U}$, which is equal to σ_U / U

- Note that relative uncertainty is a number without units
  - It is frequently expressed as a percent

Example 3.14

- **Question:** The radius of a circle is measured to be 5.00 ± 0.01 cm. Estimate the area of the circle and find the uncertainty in this estimate

- **Answer:**
  - The area is given by $A = \pi R^2$
  - The estimate of the area is $\pi (5.00)^2 \approx 8.5$ cm
  - The uncertainty of the area is

$$\sigma_A = \left| \frac{dA}{dR} \right| \sigma_R = 2\pi R \cdot \sigma_R = 10\pi (\text{cm}) \cdot 0.01 (\text{cm})$$

$$= 0.31 \text{cm}^2$$

  - So the estimate of the area of the circle can be expressed by

$$78.5 \pm 0.3 \ \text{cm}^2$$

Example 3.17

- **Question:** The acceleration for a mass down a frictionless inclined plane is given $a = g\sin\theta$
  - g is the acceleration due to gravity (the uncertainty in g is negligible)
  - θ is the angle of inclination of the plane ($\theta = 0.60 \pm 0.01\text{rad}$)

  Find the relative uncertainty in a

- **Answer:**

$$\frac{\sigma_a}{\mu_a} \approx \frac{\sigma_a}{a} = \sigma_{\ln a} = \left|\frac{\ln a}{d\theta}\right| \sigma_\theta$$

$$= \frac{g \cdot \cos\theta}{g \cdot \sin\theta} \cdot \sigma_\theta = \cot\theta \cdot \sigma_\theta$$

$$= \cot(0.6) \cdot 0.01 = 1.46 \cdot 0.01$$

$$\approx 1.5\%$$

Uncertainties for Functions of Several Independent Measurements

- If $X_1, \ldots, X_n$ are *independent* measurements whose uncertainties $\sigma_{X_1}, \ldots, \sigma_{X_n}$ are small, and if $U = U(X_1, \ldots, X_n)$ is a function of $X_1, \ldots, X_n$, then

$$\sigma_U \approx \sqrt{\left(\frac{\partial U}{\partial X_1}\right)^2 \sigma_{X_1}^2 + \left(\frac{\partial U}{\partial X_2}\right)^2 \sigma_{X_2}^2 + \cdots + \left(\frac{\partial U}{\partial X_n}\right)^2 \sigma_{X_n}^2}$$

  - This is the multivariate propagation of error formula
  - In practice, we evaluate the partial derivatives at the point $(X_1, \ldots, X_n)$

Uncertainties for Functions of Several Dependent Measurements

- If $X_1, \ldots, X_n$ are *not independent* measurements whose uncertainties $\sigma_{X_1}, \cdots, \sigma_{X_n}$ are small, and if $U = U(X_1, \ldots, X_n)$ is a function of $X_1, \ldots, X_n$, then a conservative estimate of σ_U is given by

$$\sigma_U \leq \left|\frac{\partial U}{\partial X_1}\right| \sigma_{X_1} + \left|\frac{\partial U}{\partial X_2}\right| \sigma_{X_2} + \cdots + \left|\frac{\partial U}{\partial X_n}\right| \sigma_{X_n}$$

  - Because in most cases of practical applications, the covariance between dependent measurements are unknown
  - This inequality is valid in almost all practical situations; in principle it can fail if some of the second partial derivatives of U are quite large
    - Due to the linear approximation using Taylor series

Example

- Question: Two perpendicular sides of a rectangle are measured to be $X = 2.0 \pm 0.1$ cm and $Y = 3.2 \pm 0.2$ cm. Find the absolute uncertainty in the area $A = XY$ (X and Y are known independent)

- Answer: First, we need the partial derivatives: $\frac{\partial A}{\partial X} = Y = 3.2$ and $\frac{\partial A}{\partial Y} = X = 2.0$, so the absolute uncertainty is

$$\sigma_A = \sqrt{3.2^2(0.01) + 2.0^2(0.04)} = .5122.$$

Summary

- We discussed measurement error
- Then we talked about different contributions to measurement error
- We looked at linear combinations of measurements (independent and dependent)
- We considered repeated measurements with differing uncertainties
- The last topic was uncertainties for functions of one measurement